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FREE (n,m)-GROUPS

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Abstract. In this paper we consider the notion of an (n,m)-group as a generalization of the notion of a group, and give a combinatorial description of a free (n,m)-group with a given basis.

Let $n > m > 0$. An (n,m)-semigroup $(A;f)$ is obtained by giving an associative vector-valued operation $f:A^n \to A^m$ on the underlying carrier set A. The concept of an (n,m)-group is defined similarly using associative (n,m)-operation and demanding the solvability of equations (i.e. an associative quasigroup is a group in this new context as well). In this parlance an ordinary group is simply a (2,1)-group, 2 being the arity of the basic group

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operation, say a·b, and 1 being the unary arity corresponding to the actual value c (if a·b=c). An (n,1)-group is just an n-group.

The class of all (n,m)-groups form variety and hence free objects do exist. Upto now decent examples of (n,m)-groups are not always available. The main result of this paper is an explicit combinatorial construction of the free (n,m)-group with a given basis using groups freely generated by given basis. This result is used to derive two consequences:

(1) Let k devide m. Then every (m+k,m)-group can be embedded in an (m+1,m)-group.

(2) If A and B are two equicardinal sets then the free (m+k,m)-groups generated by A and B are isomorphic.

0.Preliminaries

The notion of vector valued groups is introduced in [1]. A partial answer to the problem of satisfactory description of free vector valued groups is given in [4]. In the last two years we have completely solved this problem. The solution is a combinatorial description of free vector valued groups and is given in this paper. We are grateful to Profesor Čupona for the helpful conversations and sugestions throughout this project.

Now we introduce the basic notions and state the basic results. Let Q be a nonemty set. For a positive integer p, Q^p denotes the p-th cartesian power of Q. Instead of writing $(a_1,...,a_p)$ for an element of Q^p, we will use the notation a_1^p and/or $a_1...a_p$. With this notation we can identify Q^p with the subset $\{a_1...a_p | a_i \in Q\}$ of the free semigroup Q^+ generated by Q. (Here, $a_1 \ldots a_p$ denotes the product of $a_1, \ldots, a_p$ in Q^+.)

Let m and n be positive integers with $n-m=k\geq 1$. An (n,m)-operation on Q is a map $f:Q^n \to Q^m$. The pair $(Q;f)$ is called an (n,m)-groupoid. An (n,m)-groupoid is called (n,m)-semigroup ([1]), if for every $1\leq i\leq k$, and every $x_1^{n+k} \in Q^{n+k}$,

$$f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}). \tag{0.1}$$

An (n,m)-semigroup is called an (n,m)-group ([1]), if for each $a \in Q^k$, $b \in Q^m$, the equations

$$f(ax) = b = f(ay) \tag{0.2}$$

have solutions $x, y \in Q^m$.

Since the notion of (n,1)-groups is the same as the notion of n-groups, and specially for n=2, is the same as the notions of groups, from now on we consider (n,m)-groups, only for $n-m=k\geq 1$, $m\geq 2$, and call them vector valued (shortly v.v.) groups.

As the general associative law is valid for v.v. semigroups ([2]), we use the notation $[\,]:Q^n \to Q^m$ instead of $f:Q^n \to Q^m$ and $[a_1^{m+sk}]$ instead of $[\,]^s(a_1^{m+sk})$, where $[\,]^s$ is the (m+sk,m)-operation induced by [].

An (m+k,m)-operation [] on Q is equivalent to m n-ary operations $[\,]_1, \ldots, [\,]_m$ on Q defined by

$$(\forall i \in \mathbb{N}_m)\, [a_1^{m+k}]_i = b_i \Leftrightarrow [a_1^{m+k}] = b_1^m, \tag{0.3}$$

where $\mathbb{N}_m = \{1, 2, \ldots, m\}$.

Let Q be a nonempty subset of a given group $G=(G;\cdot)$. We denote by π the canonical mapping from Q^+ into

G defined by $\pi(a_1^t)=a_1\cdot a_2\cdot\ldots\cdot a_t$, for $t\in\mathbf{N}$. Its restriction on Q^t is denoted by π_t, and the image $\pi(Q^t)=\pi_t(Q^t)$ by Q_t. It is obvious that $Q_1=Q$, and $Q_{t+1}=Q_t\cdot Q$, where $A\cdot B=\{a\cdot b \mid a\in A, b\in B\}$, for $A,B\subseteq G$.

We say that Q is t-**free** in G if the map $\pi_t:Q^t\longrightarrow G$ is injection. Note that if Q is t-free in G, then Q is r-free in G for each $r\le t$. It is obvious that $Q\subseteq G$ is 1-free in G.

The subset Q of G is said to be (m+k,m)-**subgroup** of G if Q is m-free in G, and for each $u\in Q_k$, $\langle u\rangle\cdot Q_m=uQ_m=Q_m$.

Let Q be an (m+k,m)-subgroup of G.

Let $H=\bigcup_{\alpha\ge 1}Q_\alpha$, i.e. $H=\pi(Q^+)$, and $0\le p<k$ be a positive integer such that $m+p\equiv 0 \pmod k$. Denote by Q^{-1} the set of inverses of Q in G, i.e. $Q^{-1}=\{a^{-1} \mid a\in Q\}$. Then

Proposition 0.1. (see [3,6]) (*a*) H is a subgroup of G;

(*b*) $Q_m u=uQ_m=Q_m$, for each $u\in Q_{sk}$, $s\ge 1$;

(*b*) $Q\subseteq Q_{m+p+1}$;

(*c*) If $p>0$, then $Q^{-1}\subset Q_{m+p-1}$;

if $p=0$, then $Q^{-1}\subset Q_{m+k-1}$;

(*d*) $H=Q_m\cup Q_{m+1}\cup\ldots\cup Q_{m+k-1}$,

(*e*) Q_{m+p} is a normal subgroup of H.■

If Q generates G, i.e. if

$$G=Q_m\cup Q_{m+1}\cup\ldots\cup Q_{m+k-1}, \tag{0.4}$$

then we say that G is a **covering group** for the (m+k,m)-subgroup Q. (The subgroup H of G, above, is a covering group for Q.) If, moreover, G satisfies the condition

$$Q_i \bigcap Q_j = \emptyset,\ i,j \in \{m,m+1,\ldots,m+k-1\}, i \neq j, \tag{0.5}$$

then G is called the **universal covering group** for the (m+k,m)-subgroup Q ([6]). The universal covering group of Q will be denoted by $\mathbf{Q}^{\vee}$.

The following proposition characterizes universal covering groups of an (m+k,m)-subgroup.

Proposition 0.2.([6]) **G** is a universal covering group of an (m+k,m)-subgroup Q iff **G** is a covering group of Q and k is the least positive integer such that Q is an (m+k,m)-subgroup of **G**.■

Let Q be an (m+k,m)-subgroup of a given group $\mathbf{G}=(G;\cdot)$. If we define an (m+k,m)-operation $[\![\,]\!]$ on Q by

$$[\![a_1^{m+k}]\!] = b_1^m \Longleftrightarrow \pi(a_1^{m+k}) = \pi(b_1^m) \tag{0.6}$$

then the pair $(Q,[\![\,]\!])$ is an (m+k,m)-group. We say that $\mathbf{Q}=(Q;[\![\,]\!])$ is **induced** by the (m+k,m)-subgroup Q of **G**. Conversely, for each (m+k,m)-group $\mathbf{Q}=(Q;[\,])$ we can construct a group $\mathbf{Q}^{\vee}$,such that Q is its (m+k,m)-subgroup, and $\mathbf{Q}^{\vee}$ is the universal covering group for Q (see [2]).

1. A combinatorial construction of vector valued groups

Let $\underset{\sim}{G}=(G;\circ)$ be a given group generated by a nonempty set B and let B be m-free in G.

The norm ,$\|\,\|$, of the elements of the group **G** is defined to be the zero homomorphism from **G** into $\mathbb{N}$, i.e. $\|x\|=0$, for each $x \in G$.

We will define, by induction on α, a chain of sets $F_o, F_1, \dots, F_\alpha, \dots$ and a chain of groups $G_1, G_2, \dots, G_\alpha, \dots$, such that F_α is a subset of G_α, F_α is m-free in G_α and G_α is generated by F_α.

Let $F_o = B$ and $G_o = G$. Assume that we have already constructed the set F_α and the group G_α.

Denote by R_α the set

$R_\alpha = \{x \in G_\alpha \setminus G_{\alpha-1} \mid (\exists s \geq 1)(\exists u_1^{m+sk} \in (F_\alpha)^{m+sk})\ x = \pi(u_1^{m+sk})\} \setminus (F_\alpha)^m$.

Then put $F_{\alpha+1} = F_\alpha \bigcup (\mathbb{N}_m \times R_\alpha)$, and $S_{\alpha+1} = G_\alpha \bigsqcup (\mathbb{N}_m \times R_\alpha)^*$, i.e. $S_{\alpha+1}$ is the free product in the class of monoids. The element u of $S_{\alpha+1}$, $u \neq 1$, has a unique representation in a canonical form $u_1 \dots u_s$ where $u_i \neq 1$, and u_i and u_{i+1} are not in the same component.

Using the canonical form of the elements of the free product $S_{\alpha+1}$, we can extend the norm on the elements of $S_{\alpha+1}$ in the following way:

$\|a\|$ is defined already in G_α for each $a \in G_\alpha$.

$\|(i,x)\| = 1 + \|x\|$;

$\|u_1 \dots u_s\| = \sum_{\nu=1}^{s} \|u_\nu\|$.

We choose a special subset E of $S_{\alpha+1}$, whose elements are called **elementary reductions**, by defining u to be an element of E iff it has one of the two following forms:

(e1) $u = (1,x)\dots(m,x)$, or

(e2) $u = (i,x)\dots(m,x)x^{-1}(1,x)\dots(i-1,x)$, $i \in \mathbb{N}_m \setminus \{1\}$.

We say that an element $v\in S_{\alpha+1}$ is **reducible** iff $v=v'uv''$, where u is an elementary reduction. Otherwise we say that v is **reduced**. The set of all reduced elements of $S_{\alpha+1}$ is denoted by $G_{\alpha+1}$.

We shall define a reduction, i.e. a mapping $\varphi:S_{\alpha+1}\to G_{\alpha+1}$, by induction on norm as follows:

(0) $\varphi(x)=x,\ x\in G_{\alpha+1}$.

Let $\varphi(y)$ be defined for each y such that $\|y\|<\|x\|$, and y satisfies the following condition

$$\varphi(y)\neq y\Longleftrightarrow\|\varphi(y)\|<\|y\|. \qquad (*)$$

If x has the form $x=x'ux''$, where u is an elementary reduction of the form (e1), i.e. $u=(1,z)\dots(m,z)$ and x' is of the least possible norm, then

(1) $\varphi(x)=\varphi(x'zx'')$.

If $x=x'ux''$, x does not contain an elementary reduction of the form (e1), u is an elementary reduction of the form (e2), and x' is of the least possible norm, then

(2) $\varphi(x)=\varphi(x'x'')$.

We will give, next, some properties of the mapping φ.

1.1°. (a) φ is a well defined mapping, and the condition (*) is satisfied for each $x\in S_{\alpha+1}$.

(b) $\|\varphi(u)\|\leq\|u\|,\ u\in S_{\alpha+1}$.

(c) If $u=\pi(u_1^r)$, $u_\nu\in F_{\alpha+1}$, then there exists an $s\geq1$ and $v_1^s\in(F_{\alpha+1})^s$ such that $\varphi(u)=\pi(v_1^s)$ and $r\equiv s(\mathrm{mod}\,k)$.

(d) $\varphi(x(1,y)\dots(m,y)z)=\varphi(xyz)$, $x,z\in S_{\alpha+1}$, $y\in R_\alpha$.

(e) $\varphi(xuy)=\varphi(xy)$, where u is an elementary reduction of the form (e2), and $x,y\in S_{\alpha+1}$.

(f) $\varphi(uvw)=\varphi(u\varphi(v)w)$, $u,v,w\in S_{\alpha+1}$.

Proof: The proof of these properties is by induction on the norm of the elements of $S_{\alpha+1}$. Concidering the fact that on the right hand side of the equations (1) and (2) of the definition of φ, φ is applied on elements of $S_{\alpha+1}$ with norm less then the norm of x, φ is a well defined mapping.

Let $x\in S_{\alpha+1}$, $\varphi(x)\neq x$, and $\varphi(x)=\varphi(y)$, where $\|y\|<\|x\|$. Then, by the inductive hypothesis, we have

$\|\varphi(x)\|=\|\varphi(y)\|\le\|y\|<\|x\|$.

Using the definition of φ, and R_α, the property (*c*) can be easily proved.

To prove (*d*) we need only to concider the case when x is reducible element of the form (*e*1). Let $x=x'(1,t)\ldots(m,t)x''$, where x' has the least possible norm. Then using the inductive hypothesis, and the definition of φ, we obtain

$$\varphi(x(1,y)\ldots(m,y)z)=\varphi(x'(1,t)\ldots(m,t)x''(1,y)\ldots(m,y)z)=$$
$$=\varphi(x'tx''(1,y)\ldots(m,y)z)=\varphi(x'tx''yz)=\varphi(xyz).$$

Concidering the following cases:

1. x is reduced, and y does not contain an elementary reduction of the form (*e*1);

2. x is reduced, but y is reducible of the form (*e*1);

3. x is reducible of the form (*e*1);

4. x contains an elementary reduction of the form $(e2)$.

5. $x=x'(i,t)\dots(r,t)$,

$y=(r,t)\dots(m,t)t^{-1}(1,t)\dots(r-1,t)$.

6. $x=x'(i,t)\dots(m,t)t^{-1}(1,t)\dots(r,t)$,

$y=(r+1,t)\dots(m,t)t^{-1}(1,t)\dots(r,t)$,

where $r<i$,

the definition of φ and induction on the norm, we can easily obtain the proof of (e).

(f) is a consequence of (d) and (e).■

Using the reduction φ, we define an operation * on $G_{\alpha+1}$ by

$x*y=\varphi(xy)$.

Then:

1.2° (a) $\mathbf{G}_{\alpha+1}=(G_{\alpha+1};*)$ is a group generated by $F_{\alpha+1}$;

(b) $F_{\alpha+1}$ is m-free in $\mathbf{G}_{\alpha+1}$;

(c) $\mathbf{G}_{\alpha}$ is a subgroup of $\mathbf{G}_{\alpha+1}$.

Proof: From the construction it follows that $\mathbf{G}_{\alpha+1}$ is a monoid generated by $F_{\alpha+1}$.

Let $(i,x)\in F_{\alpha+1}$. Then

$(i,x)^{-1}=(i+1,x)\dots(m,x)x^{-1}(1,x)\dots(i-1,x)$.

Thus $\mathbf{G}_{\alpha+1}$ is a group generated by $F_{\alpha+1}$.

By the construction of $\mathbf{G}_{\alpha+1}$, it follows that $\mathbf{G}_{\alpha}\le\mathbf{G}_{\alpha+1}$.

It remains to prove the property (b), i.e. that $F_{\alpha+1}$ is m-free in $\mathbf{G}_{\alpha+1}$.

Let $u_\nu, v_\nu \in F_{\alpha+1}$, and let $u_1 * \ldots * u_m = v_1 * \ldots * v_m$ in $G_{\alpha+1}$. Then $\varphi(u_1 \ldots u_m) = \varphi(v_1 \ldots v_m)$. If both $u_1 \ldots u_m$ and $v_1 \ldots v_m$ are reducible elements of $S_{\alpha+1}$, i.e. $u_\nu = (\nu, x)$, $v_\nu = (\nu, y)$, $\nu \in \mathbb{N}_m$, $x, y \in R_\alpha$, then we have $\varphi(u_1 \ldots u_m) = x$ and $\varphi(v_1 \ldots v_m) = y$, i.e. $x = y$, hence $(\nu, x) = (\nu, y)$, $\nu \in \mathbb{N}_m$.

If $u_1 \ldots u_m$ is reduced and $v_1 \ldots v_m$ is not, then $v_\lambda = (\lambda, y)$, and $\varphi(v_1 \ldots v_m) = y \in R_\alpha$. Thus $u_1 \ldots u_m \in R_\alpha$, which contradicts the definition of R_α.

If u_ν, $v_\nu \in F_\alpha$, then the claim is true, as F_α is m-free in $G_\alpha \le G_{\alpha+1}$. It remains to concider the case when at least one u_ν is an element of $F_{\alpha+1} \setminus F_\alpha$. Then there is a $\lambda \in \mathbb{N}_m$, such that $v_\lambda \in F_{\alpha+1} \setminus F_\alpha$. Let

$$u_1 \ldots u_m = u_1 \ldots u_{j_0} (i_1, x_1) u_{j_0+2} \ldots u_{j_1} (i_2, x_2) \ldots u_{j_{s-1}} (i_s, x_s) \ldots u_{j_s}$$

and

$$v_1 \ldots v_m = v_1 \ldots v_{l_0} (i_1, x_1) v_{l_0+2} \ldots v_{l_1} (i_2, x_2) \ldots v_{l_{s-1}} (i_s, x_s) \ldots v_{l_s}$$

where $s < m$, $0 \le j_0, l_0 \le m-1$, $j_s = l_s = m$.

As $u_1 \ldots u_m$, $v_1 \ldots v_m$ are elements of the free product of monoids, it follows that

$$u_1 \ldots u_{j_0} u_{j_0+2} \ldots u_{j_1} \ldots u_{j_{s-1}} u_{j_{s-1}+2} \ldots u_{j_s} =$$
$$= v_1 \ldots v_{l_0} v_{l_0+2} \ldots v_{l_1} \ldots v_{l_{s-1}} v_{l_{s-1}+2} \ldots v_{l_s}$$

in G_α. The products on both, the left and right hand sides of the equation have m-s elements. Thus, as F_α is m-free in G_α, it is m-s-free in G_α, as well, and we obtain that $u_\lambda = v_\lambda$, for every $\lambda \in \mathbb{N}_{m-s}$. Hence we have proved that $F_{\alpha+1}$ is m-free in $G_{\alpha+1}$.■

Let $\overline{G} = \bigcup_{\alpha\geq o} G_\alpha$ and $F = \bigcup_{\alpha\geq o} F_\alpha$.

As $\overline{G}$ is a union of a chain of groups it is a group. Using induction on α, and 1.2^o, we obtain the following proposition:

1.3^o (a) $\overline{G}=(\overline{G};*)$ is a group generated by F;

(b) F is m-free in $\overline{G}$.■

Using the $1.3^o(b)$ and the operation * defined in $\overline{G}$, we can define an (m+k,m)-operation [] on F by

$$[u_1^{m+k}]=v_1^m \Leftrightarrow u_1*u_2*\ldots*u_{m+k}=v_1*v_2*\ldots*v_m. \tag{1.1}$$

Then

1.4^o (a) $\mathbf{F}=(F;[])$ is an (m+k,m)-group.

(b) F is an (m+k,m)-subgroup of $\overline{G}$, and $\overline{G}$ is its covering group.

Proof: *(a)* follows from *(b)*. Concidering 1.3^o, it remains to prove that

$(\forall u_\mu, v_\lambda \in F)(\exists x,y\in\overline{G})\, u_1*\ldots*u_k*x=v_1*\ldots*v_m,\ y*u_1*\ldots*u_k=v_1*\ldots*v_m$

such that $x=w_1*\ldots*w_m$, $y=t_1*\ldots*t_m$, for some $w_\tau, t_\tau\in F$.

Let $u_\mu, v_\lambda\in F$ be given, and take $w_\tau=(\tau, u_k^{-1}\ldots u_1^{-1}v_1\ldots v_m)$ $t_\tau=(\tau, v_1\ldots v_m u_k^{-1}\ldots u_1^{-1})$. Then $x=w_1*\ldots*w_m$, $y=t_1*\ldots*t_m$ are elements of $\overline{G}$ with the required property.■

2. Free (m+k,m)-groups

The construction given in the previous part will be used to construct a free (m+k,m)-group with a given basis A. Two cases arise. Namely, if k is a divisor of m, and if k is not a divisor of m. The construction differs only in the choice of the starting group G_o. We will first give a lemma.

Let $\mathbf{F}=(F;[])$ be the $(m+k,m)$-group obtained by the construction, such that $\mathbf{F}^{\vee}=\bar{G}$.

Theorem 2.1°. Let $\mathbf{Q}=(Q;[\![\,]\!])$ be an $(m+k,m)$-group, $\lambda:B\to Q$ a mapping and let there exist a unique homomorphism $\xi_o:G_o\to\mathbf{Q}^{\vee}$ which is an extension of λ. Then there exists a unique $(m+k,m)$- homomorphism $\xi:\mathbf{F}\to\mathbf{Q}$ which is an extension of λ.

Proof: By induction on α, we construct a sequence ξ_α of mappings from G_α into $\mathbf{Q}^{\vee}$.

Assume that a unique homomorphism $\xi_{\alpha-1}:G_{\alpha-1}\to\mathbf{Q}^{\vee}$ has already been constructed. As $S_\alpha=G_{\alpha-1}\coprod(\mathbb{N}_m\times R_{\alpha-1})^*$, there exists a unique homomorphism ξ'_α from S_α into $\mathbf{Q}^{\vee}$ which is an extension of both $\xi_{\alpha-1}$ and the homomorphism μ_α from the free monoid $(\mathbb{N}_m\times R_{\alpha-1})^*$ into $\mathbf{Q}^{\vee}$, where μ_α is the unique homomorphique extension of the mapping $\zeta_{\alpha-1}:\mathbb{N}_m\times R_{\alpha-1}\to Q^{\vee}$, defined by:

$$\zeta_{\alpha-1}(i,y)=[\![\xi_{\alpha-1}(y)]\!]_i.$$

Define ξ_α to be the restriction of ξ'_α on G_α.

Then, by induction on the norm, $\xi'_\alpha(x)=\xi'_\alpha(\varphi(x))$, which implies that ξ_α is a homomorphism from G_α into $\mathbf{Q}^{\vee}$, and by the inductive definition of ξ_α, that it is a unique one with the required properties. If we denote by ξ' the homomorphism induced by ξ_α, $\alpha\geq0$, then the restriction ξ of ξ' over F is the unique homomorphique extension of the mapping λ.■

As a consequence of this theorem and the construction, we obtain a construction of a free (m+k,m)-group with a basis A.

Theorem 2.2°(a) If k is not divisor of m, and we choose G_o to be the free group with the basis A, then the (m+k,m)-group **F** obtained by the given construction is a free (m+k,m)-group with a basis A ([8]).

(b) If k is a divisor of m, and we choose G_o to be the free product $H \sqcup C_m$, where **H** is the free group with a basis A, and C_m is a cyclic group of order m generated by $e \notin A$, then the (m+k,m)-group **F** obtained by the given construction is a free (m+k,m)-group with a basis A ([7]).■

Note that in 2.2°.(*b*) A could be an empty set as well.

The difference between the case (*a*) and (*b*) in the previous theorem arise from the fact that in (*b*) the unity e of the covering group could be written in the form $e_1 \dots e_m$, where $e_i \in F$.

Let $\mathbf{Q}=(Q;[\![\,]\!])$ be an (m+k,m)-group such that k is a divisor of m, choose G_o to be $\mathbf{Q}^{\vee}$, and using the procedure given in 1, construct an (m+1,m)-group. Then the (m+1,m)-group $\mathbf{F}=(F;[\,])$ thus obtained is such that $Q \subseteq F$, and $[\![a_1^{m+k}]\!] = [a_1^{m+k}]$, for each $a_\nu \in Q$. Thus, we have the following:

Theorem 2.3°. (Post theorem for v.v. groups when k is divisor of m [7]) Let $\mathbf{Q}=(Q;[\,])$ be an (m+k,m)-group, such that k is divisor of m. Then there exists an (m+1,m)-

group $P=(P;[\![\,]\!])$ such that $Q\subseteq P$, and for any $a_\lambda\in Q$, $\lambda\in\{1,2,\ldots,m+sk\}$, $s\geq 1$

$$[a_1^{m+sk}]=[\![a_1^{m+sk}]\!].\blacksquare$$

Finally, let us give some remarks on the cardinality of the basis of a free v.v. group.

From the construction of the free $(m+k,m)$-group **F** with a basis A, it follows that:

2.4° If the cardinality of A is infinite, then F and A have the same cardinality.■

2.5° If A and B are equivalent sets, then the free $(m+k,m)$-groups $\mathbf{F}_A$ and $\mathbf{F}_B$ with basis A and B respectively, are isomorphic.■

Using the known Fudzivara theorem ([9]) for free algebras, we obtain the following

2.6° Let $\mathbf{F}_A$ and $\mathbf{F}_B$ are isomorphic free $(m+k,m)$-groups with basis A and B repectively. If one of the basis is infinite, then A and B are equivalent.■

The open question is for the case when the basis are finite. We do not know whether isomorphic free $(m+k,m)$-groups $\mathbf{F}_A$ and $\mathbf{F}_B$ with finite basis A and B have the property that A and B are equivalent, i.e. have the same number of elements.

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