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ADDENDA TO THE MONOGRAPH "RECENT ADVANCES IN GEOMETRIC INEQUALITIES", II

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Abstract. In this paper we prove some inequalities related to the elements of a triangle. They are generalizations of the results from [3], and some are of a new nature. This paper is a continuation of [2]. We shall follow the terminology of [3].

Theorem 1. If x,y,z are arbitrary positive real numbers, then

$$\frac{x}{y+z}\cdot\frac{a^2}{r_a^2}+\frac{y}{z+x}\cdot\frac{b^2}{r_b^2}+\frac{z}{x+y}\cdot\frac{c^2}{r_c^2}=\Sigma\frac{x}{y+z}\cdot\frac{a^2}{r_a^2}\geq 2. \tag{A}$$

Proof. Applying Cauchy's inequality on the sequences of real numbers

$$(\sqrt{y+z},\ \sqrt{z+x},\ \sqrt{x+y}) \text{ and } \left(\frac{a}{r_a\sqrt{y+x}},\ \frac{b}{r_b\sqrt{z+x}},\ \frac{c}{r_c\sqrt{x+y}}\right),$$

we have

$$\Sigma\frac{x}{y+z}\cdot\frac{a^2}{r_a^2}\geq\frac{1}{2}\left(\Sigma\frac{a}{r_a}\right)^2-\left(\Sigma\frac{a^2}{r_a^2}\right). \tag{1}$$

It is known that

$$\Sigma\frac{a}{r_a}=\frac{2}{s}(4R+r) \text{ and } \Sigma\frac{a^2}{r_a^2}=\frac{2}{s^2}((4R+r)^2-s^2), \tag{2}$$

so (2) and (1) imply

$$\Sigma\frac{x}{y+z}\cdot\frac{a^2}{r_a^2}\geq 2.$$

Remark 1. This result is a generalization of (IX 8.32.(4)) in [3].

Theorem 2. If x,y,z are arbitrary positive real numbers, then

$$\Sigma\frac{x}{y+z}\cdot\frac{1}{h_a^2}\geq\frac{9}{2s^2}. \tag{B}$$

Proof. Using Cauchy's inequality, we have

$$(\Sigma\frac{1}{h_a})^2 = (\Sigma\sqrt{y+z}\cdot\frac{1}{h_a\sqrt{y+z}})^2 \le (\Sigma(y+z))(\Sigma\frac{1}{h_a^2(y+z)}),$$

which is equivalent to

$$\Sigma\frac{x}{y+z}\cdot\frac{1}{h_a^2} \ge \frac{1}{2}(\Sigma\frac{1}{h_a})^2 - (\Sigma\frac{1}{h_a^2}). \tag{3}$$

It is known that

$$\Sigma\frac{1}{h_a} = \frac{1}{r},\ \Sigma\frac{1}{h_a^2} = \frac{1}{(2F)^2}(\Sigma a^2),\ \Sigma a^2 = 2(s^2-4Rr-r^2). \tag{4}$$

Now (4) and (3) imply

$$\Sigma\frac{x}{y+z}\cdot\frac{1}{h_a^2} \ge \frac{4R+r}{2rs^2}. \tag{5}$$

Then inequality (B) is obtained using (5) and the inequality 5.1 in [1], i.e. $R \ge 2r$.

Theorem 3. In every triangle

$$\Sigma a^2\sec^3\frac{\alpha}{2} \ge 8R^2\sqrt{3}. \tag{C}$$

Equality holds if and only if the triangle is equilateral.

Proof. The following function

$$f(x) = \frac{\sin^2 x}{\cos},\qquad 0 < x < \frac{\pi}{2}$$

is convex, so

$$\sum_{i=1}^{3}\frac{\sin^2 x_i}{\cos x_i} \ge \frac{3\sin^2\frac{1}{3}(\sum_{i=1}^{3} x_i)}{\cos\frac{1}{3}(\sum_{i=1}^{3} x_i)}.$$

If we put $x_1 = \frac{\alpha}{2}$, $x_2 = \frac{\beta}{2}$ and $x_3 = \frac{\gamma}{2}$, then

$$\Sigma\frac{\sin^2\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \ge 3\cdot\frac{\sin^2(\pi/6)}{\cos(\pi/6)} = \frac{\sqrt{3}}{2}.$$

Using the sine law we get the required inequality.

<u>Theorem 4</u>. In every triangle the following inequality holds

$$\Sigma a^n \cot\frac{\beta}{2}\cot\frac{\gamma}{2} \geq 2^n \cdot s^n \cdot (R-r)^{n/2}(4R+r)^{1-(n/2)} \cdot r^{-1}, \quad (n\geq 2), \tag{D}$$

with the equality only if the triangle is equilateral and n = 2.

<u>Proof</u>. If we put in Jensen's inequality for a convex function $n=3$, $x_1=\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$, $x_2=\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}$, $x_3=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$, $p_1=a^2\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$, $p_2=b^2\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}$, $p_3=c^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$ and $f(x) = x^{n/2}(x>0,\ n\geq 2)$, from $\Sigma\cot\frac{\beta}{2}\cot\frac{\gamma}{2} = 1 + \frac{4R}{r}$ and $\Sigma a^2\cot\frac{\beta}{2}\cot\frac{\gamma}{2} = \frac{4s^2(R-r)}{r}$, we obtain the inequality (D).

<u>Theorem 5</u>. In every triangle

$$\Sigma\frac{r_a}{h_a-r} \geq \frac{9}{2}. \tag{E}$$

Equality holds if and only if the triangle is equilateral.

<u>Proof</u>. Since $r_a=F/(s-a)$, $h_a=2F/a$ and $r=F/s$, we have $r_a/(h_a-r) = as/(s-a)(2s-a))$, that is,

$$\Sigma\frac{r_a}{h_a-r} = s(\Sigma\frac{a}{(s-a)(2s-a)}). \tag{6}$$

The following function

$$f(x) = \frac{x}{(s-x)(2s-x)}, \quad 0<x<s,$$

is convex, so

$$\sum_{i=1}^{3}\frac{x_i}{(s-x_i)(2s-x_i)} \geq \frac{9(\sum_{i=1}^{3} x_i)}{(3s-\sum_{i=1}^{3} x_i)(6s-\sum_{i=1}^{3} x_i)}.$$

Let $x_1=a$, $x_2=b$ and $x_3=c$, then we obtain

$$\Sigma\frac{a}{(s-a)(2s-a)} \geq \frac{9}{2s}$$

and, in view, the required inequality also.

R E F E R E N C E S

[1] Bottema O., and oth.: Geometric inequalities, Wolters-Noordhoff, Groningen, 1969

[2] Milošević D.M.: Addenda to the monograph "Recent advances in geometric inequalities", I (to appear)

[3] Mitrinović D.S., Pečarić J.E., Volenec V.: Recent advances in geometric inequalities, Kluwer Academic Publishers, Dordrecht/Boston/London, 1989

PRILOG ZA MONOGRAFIJU "RECENT ADVANCES IN GEOMETRIC INEQUALITIES", II

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R e z i m e

U ovom radu dati su dokazi nekih nejednakosti za elemente trougla. Neke od nejednakosti predstavljaju generalizaciju rezultata iz [3], a neke su nove. Korišćena je terminologija iz