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THE CAUCHY PROBLEM FOR THE QUASILINEAR SCHRÖDINGER EQUATION AND LOCAL WELL POSEDNESS

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Abstract. In this paper we will examine the posibility of well-posedness of Cauchy problem in the case of Schrodinger nonlinear equations, i.e. Schrodinger quazi-linear equations. Equations of such type appear in several fields of physics, such as plasma fluids, classical and quantum ferromagnetism, laser theory, etc., and also in complex geometry, where, for example, in Kähler geometry they model ,,Schrödinger flows". The results of this paper apply to the superfluid film equation in fluid mechanics. The method is based on energy estimates which can be performed thanks to the construction of an integrating factor. This construction is of independent interest and relies on the analysis of some new pseudo-differential operators. These equations are also analogous to corresponding ones for hyperbolic equations, where the corresponding results were obtained much earlier, in the 70's, by Kato and his collaborators [10], [11]. The problem was extensively studied in the 90's, in the case of constant coefficients.

1. Introduction

We will consider the Schrodinger nonlinear equations on a compact Riemannian manifold (M, g)

$$i\partial_t u + \Delta_g u = f\left(|u|^2\right) u, \quad u(0,x) = u_0(x) \tag{1}$$

where f is a suitably chosen real valued function, while Δ is the Laplace operator. A natural question is whether the particular structure of (M, g) influences the critical threshold for the local well-posedness Sobolev regularity of the initial data. Let us precise what we call local well-posedness.

Definition 1. *([4]) We say that the Cauchy problem is locally well-posed for data in the Sobolev space $H^s(M)$ if for any $R > 0$ there exist $T > 0$ and a functional space X_T continuously embedded in $C([-T,T], H^s(M))$ and invariant under the natural action of the isometrics of M, such that for every*

$$u_0 \in B_R = \left\{u_0 \in H^s(M) \mid \|u_0\|_{H^s(M)} < R\right\}$$

the Cauchy problem (1) has a unique solution $u \in X_T$. Moreover:
1. The map $u_0 \to u$ is uniformly continuous from B_R to $C([-T,T], H^s(M))$.

2. If $u_0 \in H^1(M)$, $u \in C\left([-T,T], H^1(M)\right)$ *and satisfies the usual conservation laws:*

$$\|u(t)\|_{L^2(M)} = \|u_0\|_{L^2(M)},$$
$$\|\nabla_g u(t)\|_{L^2(M)} + \int_M F\left(|u(t,x)|^2\right)dx = const,$$

where F *is a primitive of* f.

The assumption of uniform continuity of the flow map in the above definition seems to be natural for semilinear equations. The main issue in our analysis is to study the nonlinear evolution by the nonlinear Schrodinger equation flow of some eigenfunctions of the Laplace-Beltrami operator on the d dimensional sphere S^d. The situation turns out to be particularly simple in $1D$. Consider the Cauchy problem:

$$i\partial_t u + \partial_x^2 u = |u|^2 u, \; u(0,x) = u_0(x), \tag{2}$$

where $x \in S^1$, $t \in R$. For $s < 0$ we set

$$u_{k,n}(t,x) = kn^{-s}\exp\left(-it\left(n^2 + k^2 n^{-2s}\right)\right)\exp(inx).$$

It is easy to check that $u_{k,n}$ solves (2) with initial data $kn^{-s}\exp(inx)$. Moreover

$$\|u_{k,n}(t,*)\|_{H^s(S^1)} \leq |k|,$$

where k is a fixed element of the interval $(0,1)$. Let $\{k_n\}$ be a sequence of real numbers tending to k which will be specified later. We observe that $u_{k,n}(0,*) - u_{k_n,n}(0,*) \to 0$ in $H^s\left(S^1\right)$ as $n \to \infty$. Take now a positive t. Then there exist $C > 0$ independent of n and $\delta > 0$ such that

$$\|u_{k,n}(t,*) - u_{k_n,n}(t,*)\|_{H^s(S^1)} \geq C\left|\exp\left(-itn^{-2s}\left(k^2 - k_n^2\right)\right) - 1\right| - Cn^{-\delta}. \tag{3}$$

If we suppose that (2) is locally well-posed in $H^s\left(S^1\right)$, $s < 0$, then (3) would imply

$$\lim_{n\to+\infty}\left|\exp\left(-itn^{-2s}\left(k^2 - k_n^2\right)\right) - 1\right| = 0 \tag{4}$$

But (4) easily fails by choosing $\{k_n\}$, so that

$$\left(k^2 - k_n^2\right)n^{-2s} = \alpha n^\beta$$

for suitable $\alpha > 0$, $\beta > 0$, satisfying $2s + \beta < 0$.

Theorem 1. *Let* $s < 0$. *Then the Cauchy problem (2) is not locally well-posed for data in* $H^s\left(S^1\right)$.

Remark 1: 1. It can be shown that (2) is locally well-posed for data in $H^s(S^1)$, $s \geq 0$.

2. The proof of Theorem 1 can be extended to equations of type

$$i\partial_t u + \partial_x^2 u = f\left(|u|^2\right)u \tag{5}$$

under weak assumptions on the nonlinearity, for example, $f(\lambda) = \pm\lambda^\gamma$ for some $\gamma > 0$. In the case of (5) one has to deal with the following explicit solution

$$u_{k,n}(t,x) = kn^{-s}\exp\left(-it\left(n^2 + f\left(k^2 n^{-2s}\right)\right)\right)\exp(inx).$$

3. It is interesting to mention that if the measure invariance (if u is a solution of (5) then so is $e^{i\theta}u$, $\theta \in R$) of (5) is violated then one can obtain the local well-posedness of the corresponding Cauchy problem for data more singular than $L^2\left(S^1\right)$. For instance, the Cauchy problem associated to the equation

$$i\partial_t u + \partial_x^2 u = u^2$$

is locally well-posed for data in $H^s\left(S^1\right),\ s > -\frac{1}{2}$([18]).

4. A result related to Theorem 1, due to Kenig-Ponce-Vega, when the spatial domain is R is obtained in [4]. It is shown that the cubic focusing nonlinear Schrodinger equation in 1D, posed on R, is locally ill-posed for data in $H^s(R)$, $s < 0$.

We now turn to the higher dimensional case. Let (M, g) be a two dimensional Riemannian manifold and Δ_g let be the a corresponding Laplace-Beltrami operator. Now we consider the Cauchy problem

$$i\partial_t u + \Delta_g u = |u|^2 u,\ u(0,x) = u_0(x), \tag{6}$$

where $x \in M$, $t \in R$.
Let $M = R^2$, with the plain metric. Then (6) is invariant by a scaling transformation. Namely, if $u(t,x)$ is a solution of (6) then so is

$$u_\lambda(t,x) = \lambda u\ \left(\lambda^2 t, \lambda x\right)$$

with initial data $\lambda u_0(\lambda x)$. Clearly $\lambda u_0(\lambda x)$ has the same norm in $L^2\left(R^2\right)$ as $u_0(x)$. Heuristically this scaling argument suggests that (6) is locally well-posed for data in $H^s(M),\ s > 0$. Moreover this is the case when $M = R^2$ or $M = T^2$, with the plain metrics, due to [3] in the case R^2 and [1] in the periodic case. In this paper we will show that the above heuristics fail when $M = S^2$.

Theorem 2. *Let $T > 0, s \in \left]\frac{3}{20}, \frac{1}{4}\right[, k \in]0,1[$. Take $M = S^2$ with the canonical metric in (6). For $n \in N$, we denote by $\psi_n : S^2 \to C$ the restriction to S^2 of the harmonic polynomial $(x_1 + ix_2)^n$. Then for $t \in [0,T]$ the solution $u_n(t)$ of (6) with initial data $k\phi_n$, where $\phi_n = n^{\frac{1}{4}-s}\psi_n$ can be represented as*

$$u_n(t) = k\exp\left(-it\left(n(n+1) + k^2\omega_n\right)\right)\left(\phi_n + r_n(t)\right), \tag{7}$$

where $\omega_n \approx n^{\frac{1}{2}-2s}$ and $r_n(t)$ satisfies

$$\|r_n(t)\|_{H^s(S^2)} \le C_T n^{-\delta} \tag{8}$$

where $\delta > 0$ and C_T depends on T but not on n. Moreover there exists $C > 0$, independent of T and n such that

$$\|u_n\|_{L^\infty(R;H^s(S^2))} \le C_k. \tag{9}$$

As a consequence the Cauchy problem (6) is not locally well-posedness for data in $H^s\left(S^2\right)$.

Remark 2: 1. The existence of $u \in C^\infty\left(R \times S^2\right)$ is guaranteed by Theorem 2 of [2].

2. The condition $s > \dfrac{3}{20}$ ensures that (7) is valid on an arbitrary time interval. If one is interested only in the local well-posednessness, a slight modification of the proof of Theorem 2 gives the ill-definedness of the Cauchy problem (6) in $H^s\left(S^2\right), \ s \in \left[0, \frac{1}{4}\right[$.

3. In [4], we have proved the local well-posedness of (6) in $H^s(M), \ s > \frac{1}{2}$. Therefore, in the case $M = S^2$ the critical Sobolev regularity for the local well-posedness of (6) is in the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$.

4. The choice of ψ_n is related to earlier works on spherical harmonics by Stanton-Weinstein [19] and Sogge [6]. In these references, is proven that ψ_n maximizes the quotient $\|\psi\|_{L^4} / \|\psi\|_{L^2}$ among the spherical harmonics of n degree. Moreover, observe that ψ_n is concentrated on the elliptical closure $x_1^2 + x_2^2 = 1$.

The method of proof of Theorem 2 can be further exploited in order to prove ill-posedness results in the energy space for H^1 subcritical nonlinear Schrodinger equation posed on S^6. On the other hand, in the H^1 supercritical case some semilinear wave equations on M^3 are not regullary in the energy space (see recent works of Brenner-Kumlin [20] and Lebeau [5]).

Theorem 3. *Let $\alpha \in \left]0, 1\right]$. Then the Cauchy problem*

$$i\partial_t u + \Delta_{S^6} u = \langle u \rangle^{\alpha} u, \quad u(0,x) = u_0(x), \tag{10}$$

where $x \in S^6$, $t \in R$, $\langle u \rangle = \sqrt{1 + |u|^2}$ is not locally well-posed for data in $H^1\left(S^6\right)$.

Remark 3: The question of extending our results to more general geometries than the sphere is still open. However, if (M, g) is a compact orientable d-dimensional Riemannian manifold with a closed stable (elliptic) geodesic, then by considering quasimodes for initial data as constructed in [21], one easily shows that for $s < \frac{d-1}{4}, d \geq 2$ the Picard iteration scheme applied to the integral formulation of the nonlinear Schrodinger equation

$$i\partial_t u + \Delta_g u = \pm |u|^2 u$$

sends us at the second iteration any ball of $H^s(M)$ into an unbounded set (see [22]).

2. The Cauchy Problem for the Quasilinears Schrödinger Equations

We will discuss in time the local well-posedness of the Cauchy problem for quasi-linear Schrodinger equations

$$\begin{cases} \partial_t u = i a_{lk}(x,t;u,\overline{u},\Delta_x u,\Delta_x \overline{u})\partial^2_{xlxk} u \ + i b_{lk}(x,t;u,\overline{u},\Delta_x u,\Delta_x \overline{u})\partial^2_{xlxk} \overline{u} \\ +\overrightarrow{b_1}(x,t;u,\overline{u},\Delta_x u,\Delta_x \overline{u})\Delta_x u \ + \overrightarrow{b_2}(x,t;u,\overline{u},\Delta_x u,\Delta_x \overline{u})\Delta_x \overline{u} & x \in R^n, t \in [0,T] \\ +c_1(x,t;u,\overline{u})u + c_2(x,t;u,\overline{u})\overline{u} \ + f(x,t) \\ u|_{t=0} = u_0 \end{cases} \tag{11}$$

We will determine in acceptable way ellipticity hypotheses on a_{lk}, b_{lk}, smoothness on all the coefficients, "asymptotic-flatness" on the coefficients, and as we shall see a (necessary) "non-trapping" condition on a Hamiltonian flow obtained from the coefficients and the data u_0. By local well-posedness in a space B, we mean that, given $u_0 \in B,\ f \in X$, there exists $T = T(u_0, f)$, and a unique $u \in C([0,T];\, B)$, such that u solves the equation (in a suitable sense), $u\,(0,*) = u_0$, and the mapping $(u_0, f) \in B \times X \to u \in C([0,T];\, B)$ is continuous. In general, the space B will be a Sobolev space, like

$$H^S(R^n) = \left\{ f \in S^{'} \mid \int (1+|\xi|^2)^S \left| \widehat{f}(\xi) \right|^2 d\xi < \infty \right\} \tag{12}$$

or of the type $H^S(R^n) \cap L^2(|x|^N\, dx)$, whose presence will be explained later on. It turns out that the classical theory of pseudo-differential operators is an appropriate and useful tool in this task, and we will review it and utilize it.
We take into consideration the case of constant coefficients ((x,t) independed) and semilinear non-linearity, so

$$\left\{ \begin{array}{ll} \partial_t u = i\Delta u + F(u, \overline{u}, \Delta_x u,\ \Delta_x \overline{u}) & \\ u|_{t=0} = u_0 & ,\quad x \in R^n\ ,\ t \in [0,T] \end{array} \right. , \tag{13}$$

Let us first discuss (13) in the case when there are no derivatives in the non-linearity, when $F(u,\ \overline{u},\ \Delta_x u,\ \Delta_x \overline{u}) = G(u, \overline{u})$ with $G(0,0) = 0$. In this case, the energy method applies, and gives local-well-posednessness in $H^S(R^n)$ for $s > n/2$. Since the energy method will be important to us in the sequel, let us work out this result:
Thus, we assume $G(0,0) = 0,\ G \in\ C^\infty(C \times C)$, and we wish to show the local well-posedness of the Cauchy problem

$$\left\{ \begin{array}{l} \partial_t u = i\Delta u + G(u, \overline{u}) \\ u|_{t=0} = u_0 \end{array} \right. \tag{14}$$

in the Sobolev space $H^S(R^n)$ for $s > n/2$. To simplify the exposition, let us assume that G is a polynomial, so that

$$G(u, \overline{u}) = \sum_{\substack{0 \le j \le M \\ 0 \le k \le N \\ (j,k) \ne (0,0)}} c_{jk} u^j u^{-k} \tag{15}$$

Let us recall some properties of Sobolev space:
1° $\|u\|_{L^\infty(R^n)} \le C\, \|u\|_{H^S(R^n)}$ for $s > n/2$.
2° For $s > n/2$, $H^S(R^n)$ is an algebra under pointwise multiplication. So $\|f \cdot g\|_{H^S} \le\ C\ \|f\|_{H^S}\ \|g\|_{H^S}$
This is a consequence of property 1°.
3° For $s > n/2$, if $G(0,0) = 0$, G is smooth, then

$$\|G(u, \overline{u})\|_{\ H^S} \ \le\ R\ (\|u\|_{H^s})$$

where R is an increasing function that depends on $G,\ s$, with $R(0) = 0$. For instance, in our polynomial case, we have

$$\|G(u, \overline{u})\|_{\ H^s} \ \le\ C \left\{\ \|u\|_{H^S}^{M+N} + \|u\|_{H^S} \right\} .$$

At the beginning we do the nessecarry calculations to determine the excistence and unicity. Assume that u is a regular solution of (14). Let $\hat{J}^s u\,(\xi,t) = \hat{u}(\xi,t)(1+|\xi|^2)^{s/2}$. We take (14) and rewrite it as

$$\partial_t u = i\Delta u + G(u,\overline{u}) \tag{16}$$

$$\partial_t \overline{u} = -i\Delta \overline{u} + \overline{G(u,\overline{u})}. \tag{16'}$$

We now apply J^s to both equations, multiply (16) by $\overline{J^s u} = J^S\overline{u}$, multiply (16') by $J^s u$ integrate both equations in x, and add. Then :

$$\partial_t \int |J^s u|^2 = i\int [\Delta J^s u\overline{J^s u} - \Delta\overline{J^s u}J^s u] + \int J^s G(u,\overline{u})\overline{J^s u} + \int J^s\overline{G(u,\overline{u})}J^s u$$

Since $i\int [\Delta J^s u\overline{J^s u} - \Delta\overline{J^s u}J^s u] = 0$, this term drops out. Using property $3°$, for

$$f(t) = \|u(*,t)\|^2_{H^s} = \|J^s u(*,t)\|^2_{L^2}\,,$$

we obtain

$$\begin{aligned}\left|\frac{d}{dt}f(t)\right| \;&\leq\; 2\left\|J^s G(u,\overline{u})\right\|_{L^2}\|J^s u\|_{L^2} \leq \\ &C\left\{\,\|u\|^{M+N}_{H^s} + \|u\|_{H^s}\right\}\|u\|_{H^s} \leq C\{\,f(t) \,+\, f(t)^{(M+N+1)/2}\}\end{aligned}$$

Now, we define $f_1(t) = \sup\limits_{0<r<t}\|u(*,r)\|^2_{H^s}$. Then, $\exists r_0,\ 0 \leq r_0 \leq t$ such that

$$f_1(t) = f(r_0) = \int_0^{r_0} f'(r)dr \;+\; f(0) \;\leq\; \|u_0\|^2_{H^s} \;+\; Ctf_1(t) \;+\; Ctf_1(t)^\alpha$$

where $\alpha = \frac{M+N+1}{2} > 1$. For $t \leq \frac{1}{2C}$, we obtain $f_1(t) \leq 2\|u_0\|^2_{H^s} + 2Ctf_1(t)^\alpha$. Let T_o be the first values wich satisfies $t \leq \frac{1}{2C}$ for which value $f_1(T_0) = 4\|u_0\|^2_{H^s}$. Since $f_1(t)$ is continuous,

$$4\|u_0\|^2_{H^s} \;=\; f_1(T_0) \;\leq\; 2\|u_0\|^2_{H^s} \;+\; 2CT_0 4^\alpha\,\|u_0\|^{2\alpha}_{H^s}$$

and so $T_0 \;\geq\; \dfrac{1}{C4^\alpha\|u_0\|^{2\alpha-2}_{H^s}}$. In other words, if $T_0 = \min\left\{\dfrac{1}{2C}, \dfrac{1}{C4^\alpha\|u_0\|^{2\alpha-2}_{H^s}}\right\}$, then for we have $\|u(*,t)\|^2_{H^s} \leq 4\|u_0\|^2_{H^s}$, as required.

Remark 4: Suppose we considered solutions to

$$\left\{\begin{array}{l}\partial_t = -e\Delta^2 u + i\Delta u + G(u,\overline{u}) \\ u|_{t=0} = u_0\end{array}\right. \qquad e > 0 \tag{17}$$

Then, the same conclusion holds, with C independent of e. In fact, we only need to understand $e\int [\Delta^2 J^s u\cdot\overline{J^s u} + \Delta^2\overline{J^s u}\cdot J^s u]dx = 2e\int|\Delta J^s u|^2 \geq 0$. But then

$$\begin{aligned}\partial_t \textstyle\int |J^s u|^2 = -2e\,\int|\Delta J^s u|^2 + \int J^s G(u,\overline{u})\,\overline{J^s u} + \int\overline{J^s G(u,\overline{u})}J^s u \;\leq \\ \textstyle\int J^s G(u,\overline{u})\,\overline{J^s u} + \int\overline{J^s G(u,\overline{u})}\,J^s u\end{aligned}$$

and we proceed as before.

Existence of solutions. For each $e > 0$, a solution u^e on $[0, T_e]$ to (17) is shown by "standard parabolic theory". Specifically, let $s > n/2$, and define $X_{T,M_0} = \{v : R^n \times [0,T] \to C([O,T]; H^s),\ v(0) = u_0, \text{ and } \||y|\|_T = \sup_{[0,T]} \|v(t)\|_{H^s} \le M_0\}$. We then have: for any $u_0 \in H^s$, $\|u_0\|_{H^s} \le M_0/2$, there exists $T_e = 0(e)$, depending only on M_0, s, n, G, and have a unique solution u^e in X_{T_e,M_0} of system

$$\begin{cases} \partial_t u = -e\Delta^2 u + i\Delta u + G(u,\overline{u}) \\ u|_{t=0} = u_0 \end{cases} \tag{18}$$

so that $\sup_{t\in[o,T_e]} \|u^e(t)\|_{H^s} \le M_0$. This is proved by converting (18) into the integral equation $\Gamma u^e = u^e$, where

$$\Gamma\omega(t) = e^{-et\Delta^2}u_0 + \int e^{-e(t-t')\Delta^2}[i\Delta\omega + G(\omega,\overline{\omega})]dt' \tag{19}$$

and showing that, for appropriate T_e, Γ is a contraction on $X_{T_e.M_0}$. The only estimate that is needed for the semigroup $\{e^{-et\Delta^2}, t \ge 0\}$ is that $\left\|\Delta e^{-et\Delta^2} g\right\|_{L^2} \le \dfrac{1}{e^{1/2}\,t^{1/2}} \|g\|_{L^2}$.

Set $M_0 = 8\|u_0\|_{H^s}$ Obtain, as above, a solution u^e to (18) on $[0, T^e]$. One then uses the *a priori* estimate in Remark 4, to conclude that, if $T_e \le T_0 = \min\{\dfrac{1}{2C}, \dfrac{1}{C4^\alpha \|u_0\|_{H^s}^{2\alpha-2}}\}$, one has $\sup_{t\in[o,T_e]} \|u^e(t)\|_{H^s} \le 4\|u_0\|_{H^s} \le \frac{M_0}{2}$. We can then iterate this local existence result, in the interval $[T_e, 2T_e]$, etc., to find now a solution of (18) in $[0, T_0]$, $0 < e < 1$, with $\sup_{t\in[o,T_e]} \|u^e(t)\|_{H^s} \le 4\|u_0\|_{H^s}$. Now consider $0 < e' < e < 1$, and $u^e, u^{e'}$ let be the corresponding solutions of (18). Set $v = u^e - u^{e'}$, so that

$$\partial_t v = -(e-e')\Delta^2 u^e - e'\Delta^2 v + i\Delta v + [G(u^e, u^{-e}) - G(u^{e'}, u^{-e'})]. \tag{20}$$

Recall that $\sup_{t\in[o,T_e]} \|u^e(t)\|_{L^\infty} \le M_0$ and similarly for $u^{e'}$, and that $|G(u^e, u^{-e}) - G(u^{e'}, u^{-e'})| \le C_{M_0}|u^e - u^{e'}|$. Then, multiply (20) by $\overline{v}$, conjugate (20) and multiply by v, add, and integrate in x, to obtain

$$\partial_t \int |v|^2 \le 2(e-e')||\Delta^2 u^e||_{L^2}||v||_{L^2} + C_{M_0}||v||^2_{L^2}$$

that, for $s > 4$,

$$\sup_{0<t<T} ||v||^2_{L^2} \le C(e-e')||v||_{L^2} + TC_{M_0} \sup_{0<t<T} ||v||^2_{L^2}.$$

Selecting $T \le T_0$ such that $TC_{M_0} < \frac{1}{2}$ and using that $||v||_{L^2} \le C$, we have $v \to 0$ in $C([0,T]; L^2)$ as e, $e' \to 0$ for $u^e \to u$ in $C([0.T]; L^2)$ when $e \to 0$. The family u^e belongs to $L^\infty([0,T]; H^s)$ and thus, by weak compactness, $u \in L^\infty([0,T]; H^s)$.

By the interpolation inequality

$$||v||_{H^{s-1}} \leq ||v||_{L^2}^{1/s} ||v||_{H^s}^{(s-1)/s}$$

we have

$$u \in L^\infty([0,T]; H^s) \cap C([0,T]; H^{s-1}).$$

Uniqueness. We act now like as in existence case, with $v = u - u'$, and $e = e' = 0$, where u and u' are solutions. Then obtain

$$\sup_{0<t<T} ||v||_{L^2} \leq TC_{M_0} \sup_{0<t<T} ||v||_{L^2}$$

which yields uniqueness, by taking $T \leq 1/(2C_{M_0})$.
Convergence. $u \in C([0,T]; H^s)$ depends continuously on u_o. Here there is a standard argument, due to Bona-Smith [7]. One solves with data $u_0^\delta = \varphi_\delta * u_0$, where $\varphi \in S(R^n)$, $\int \varphi = 1$, $\int x^\alpha \varphi(x) dx = 0 \forall |\alpha| \neq 0$. We then show that u^δ, the solution corresponding to u_0^δ converges in $L^\infty([0.T_0]; H^s)$, to u as $\partial \to 0$. To see this, we show

$$\sup_{[0,T_0]} ||u^\delta(t)||_{H^{s+l}} \leq C\delta^{-l} \ , \quad l > 0$$

and then use interpolation and the fact that

$$\sup_{[0.T_0]} ||(u^\delta - u^{\delta'})(t)||_{L^2} \leq C||u_0^\delta - u_0^{\delta'}||_{L^2} = o(\delta^s)$$

This completes our outline of the energy method applied to (13).

Remark 5: For power non-linearities, $G(u, \bar{u}) = |u|^\alpha u$, more refined results can be obtained by means of mixed norm estimates (the so-called Strichartz estimates). Now we briefly turn to the case of $F(u, \overline{u}, \Delta_x u, \Delta_x \overline{u})$ and explain what the energy method gives in this case. Suppose that for any $u \in H^s(R^n)$, $s > \frac{n}{2} + 1$,

$$\left| \sum_{|\alpha| \leq s} \int_{R^n} \partial_x^\alpha F(u, \overline{u}, \Delta_x u, \Delta_x \overline{u}) \partial_x^u \overline{u} dx \right| \leq C(1 + ||u||_{H^s}^\rho) ||u||_{H^s}^2 \tag{21}$$

Then the above proof works (here $\rho = \rho(F) \in$ N). Thus, for these examples, the energy method gives local well-posedness in H^s , $s > n/2 + 1$.

Example 1. $n = 1$, $F = \partial_x(|u|^k u)$.
2. $n \geq 1$, $F(u, \overline{u}, \Delta_x \overline{u})$
3. $n \geq 1$, F general, $\partial_{\partial_{x_j u}} F$, $\partial_{\partial_{x,h,\overline{u}}} F$, $j = 1, ..., n$, are real.
These results are due to Tsutsumi-Fukuda [16], Klainerman [12], Klainerman-Ponce [13], Shatah [17]. The difficulty comes from trying to "recover" the "extra-derivative" in the non-linear term.

Problem 1. Using the method of ,,artificial viscosity" and the previous estimate to prove the excistence and unicity, and the Bona-Smith method we can prove the dependness of continuity and to prove the local well-posednessness of Chauchy problem

$$\begin{cases} \partial_t u = i\Delta u + F(u, \overline{u}) \\ \partial|_{t=0} = u_0 \in H^s \quad s > \frac{n}{2} \end{cases} . \tag{22}$$

Proof. We will give an elaboration of the Bona-Smith method, which appears as Step 4 in the notes, and on the other hand we will instead work with the equation

$$\partial_t(u) = i\partial_x^2 u + iu\partial_x \overline{u}$$

in order to underscore the wide range of applicability of the method. First, we apply ∂_x^k for $k \geq 3$

$$\partial_t \int_x |\partial_x^k u|^2 = 2\mathrm{Re} i \int u\partial_x^{k+1}\overline{u}\partial_x^k\overline{u} + \text{lower order terms} \tag{23}$$

(we shall drop the lower order terms in the rest of the exposition). Pairing with $\partial_x^k \overline{u}$, integrating in x, and taking the real part gives

$$\partial_t \int_x ||\partial_x^k u|^2 = 2\mathrm{Re} i \int u\partial_x^{k+1}\overline{u}\partial_x^k\overline{u} = -\mathrm{Re} i \int_x \partial_x u(\partial_x^k \overline{u})^2. \tag{24}$$

If $||u_0||_{H^k} \leq R$, we can integrate in time to obtain $T = T(R) > 0$ for which $\sup\limits_{[0,T]} ||u(t)||_{H^k} \leq 2||u_0||_{H^k}$ is bounded. Thus existence and uniqueness of a solution on $[0,T]$ for this equation follows by the techniques of step for existence of solutions, uniqueness and convergence. Now we use the Bona-Smith method to show that the "data to solution" map is continuous as a map from H^k to $\sup\limits_{[0,T]} ||u(t)||_{H^k} \leq 2||u_0||_{H^k}$. Set $u_0^\delta = \varphi_\delta * u_0$ where $\varphi \in S(R^n)$, $\int \varphi = 1$, $\int x^\alpha \varphi(x)dx = 0$ for $|\alpha| \neq 0$. (We take φ when replacing $\hat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$, $\hat{\varphi}(\xi) = 1$, $\hat{\varphi}(\xi) = 0$ for $|\xi| \geq 2$.) Then let u^δ be the corresponding solution for u_0^δ We will treat the problem in several steps:

<u>Step I.</u> For $l \geq 0$, $\sup\limits_{[0,T]} ||u^\delta(t)||_{H^{k+l}} \leq 2R\delta^{-l}$. This is obtained from (23), (24) with k replaced by $k+l$ and also noting that $||u_o^\delta||_{H^{k+l}} \leq \delta^{-l}||u_0||_{H^k}$.

<u>Step II.</u> $\sup\limits_{[0,T]} ||(u^\delta - u)(t)||_{L^2} \leq 2||u_0^\delta - u_0||_{L^2} \leq \delta^k h(\delta)$, where $h(\delta) \to 0$ and $|h(\delta)| \leq R$. From the equation, $\partial_t(u^\delta - u) = i\partial_x^2(u^\delta - u) + iu^\delta\partial_x u^{-\delta} - iu\partial_x\overline{u} = i\partial_x^2(u^\delta - u) + i(u^\delta - u)\partial_x u^{-\delta} + iu\partial_x(\overline{u^\delta - u})$ pairing with $\overline{u^\delta - u}$, integrate in x, take the real part, integrate in time to obtain: $||(u^\delta - u)||_{L_x^2}^2 \leq ||u_0^\delta - u_0||_{L^2}^2 + T(||\partial_x u^\delta||_{L_T^\infty L_x^\infty} + ||\partial_x u||_{L_T^\infty L_x^\infty})||u^\delta - u||_{L_T^\infty L_x^2}^2$ where, to estimate the last nonlinear term, we used that $\partial_x(\overline{u^\delta - u})(\overline{u^\delta - u}) = \frac{1}{2}\partial_x(\overline{u^\delta - u})^2$ and integration by parts. Thus, by suitable choice of $T = T(R) > 0$,

$$||(u^\delta - u)(t)||_{L_x^2} \leq 2||u_0^\delta - u_o||_{L^2}.$$

Now observe

$$|\hat{\varphi}(\delta\xi) - 1| \leq \delta|\xi| \sup_{[0,\delta\xi]} |(\partial_\xi\hat{\varphi})(\eta)|$$

However, because $\partial_\xi\hat{\varphi}(0) = 0$, we also have,

$$|\partial_\xi\hat{\varphi}(\eta)| \leq \delta|\xi| \sup_{[0,\delta\xi]} |\partial_\xi^2\hat{\varphi}(\eta)|$$

Continuing, for any integer k , we have

$$|\hat{\varphi}(\delta\xi)-1|\leq\delta^{k}|\xi|^{k}\sup_{[0,\delta\xi]}|(\partial_{\xi}^{k}\hat{\varphi})(\eta)|$$

and thus

$$\left(\int|\hat{\varphi}(\delta\xi)-1|^{2}|\hat{u}_{0}(\xi)|^{2}d\xi\right)^{1/2}\leq\delta^{k}\underbrace{\left(\int\sup|\partial_{\xi}^{k}\hat{\varphi}(\eta)|^{2k}|\hat{u}_{o}(\xi)|^{2}d\xi\right)^{1/2}}_{h(\delta)}$$

$\lim_{\delta\to0}h\left(\delta\right)=0$ by dominated convergence.

Step III. For $r\leq k$, $\sup_{[0,T]}||(u^{\delta}-u)(t)||_{H^{r}}\leq R^{\frac{r}{k}}\delta^{k-r}h(\delta)^{\frac{k-r}{k}}$. This follows from Step II by interpolation:

$$||u^{\delta}-u||_{H^{r}}\leq||u^{\delta}-u||_{L^{2}}^{\frac{k-r}{k}}||u^{\delta}-u||_{H^{k}}^{\frac{r}{k}}\ .$$

Step IV. $\sup_{[0,T]}||(u^{\delta}-u)(t)||_{H^{k}}\leq2||u_{0}^{\delta}-u_{o}||_{H^{k}}$. By (23) for u and u^{δ} $\partial_{t}\partial_{x}^{k}(u^{\delta}-u)=i\partial_{x}^{2}\partial_{x}^{k}(u^{\delta}-u)+iu^{\delta}\partial_{x}^{k+1}u^{-\delta}-iu\partial_{x}^{k+1}\overline{u}\ =\ i\partial_{x}^{2}\partial_{x}^{k}(u^{\delta}-u)+i(u^{\delta}-u)\partial_{x}^{k+1}u^{-\delta}+iu\partial_{x}^{k+1}\overline{(u^{\delta}-u)}$ and thus

$$\begin{aligned}||\partial_{x}^{k}(u^{\delta}-u)(t)||_{L_{x}^{2}}^{2}\leq{}&||\partial_{x}^{k}(u_{0}^{\delta}-u_{0})||_{L^{2}}^{2}\\&+T||u^{\delta}-u||_{L_{T}^{\infty}L_{x}^{2}}||\partial_{x}^{k+1}u^{\delta}||_{L_{T}^{\infty}L_{x}^{\infty}}||\partial_{x}^{k}(u^{\delta}-u)||_{L_{T}^{\infty}L_{x}^{2}}\\&+T||\partial_{x}u||_{L_{x}^{\infty}}||\partial_{x}^{k}(u^{\delta}-u)||_{L_{x}^{2}}^{2}\end{aligned}$$

Further, using Step I and III, we estimate the first nonlinear piece (with $r=0$ in this case) to obtain

$$||\partial_{x}^{k}(u^{\delta}-u)(t)||_{L_{x}^{2}}\leq2||\partial_{x}^{k}(u_{0}^{\delta}-u_{0})||_{L^{2}}+TR\delta^{k-2}h(\delta)$$

Step V. If both $||u_{10}||_{H^{k}}\leq R$ and $||u_{20}||_{H^{k}}\leq R$, then $\sup_{[0,T]}||(u_{1}^{\delta}-u_{2}^{\delta})(T)||_{H^{k}}\leq2||u_{10}-u_{20}||_{H^{k}}$ whre $T=T(R)$. This follows by the above techniques.

We can now complete the argument. Let $\varepsilon>o$, and suppose u_{10} and u_{20} are such that $||u_{10}||_{H^{k}}\leq R$, $||u_{20}||_{H^{k}}\leq R$ and $||u_{10}-u_{20}||_{H^{k}}\leq\frac{\varepsilon}{10}$. Then obtain $\delta=\delta(u_{10},u_{20})$ such that $||u_{10}^{\delta}-u_{10}||_{H^{k}}\leq\frac{\varepsilon}{10}$ and $||u_{20}^{\delta}-u_{20}||_{H^{k}}\leq\frac{\varepsilon}{10}$. Let $T=T\left(R\right)$ (independent of δ) be such that the claims in Steps I-V hold; then the results of Steps I-V give that

$$\sup_{[0,T]}||(u^{1}-u^{2})(t)||_{H^{k}}\leq\varepsilon.$$

Problem 2. Give the proof of local well-posedness for

$$\left\{\begin{array}{l}\partial_{t}u=i\Delta u+F(u,\overline{u},\Delta_{x}\overline{u})\\ u|_{t=0}=u_{0}\ \in\ H^{s}(R^{n})\end{array}\right.$$

for $s>\dfrac{n}{2}+1$.

Proof. The following proof it seems to apply only for $k > \frac{n}{2}+2$. In the presentation, I shall restrict to the case $s = k$ integer and to $n = 1(1-D)$, and also to monomial nonlinearity, i.e.

$$F(u, \overline{u}, \partial_x \overline{u}) = u^{\alpha} u^{-\beta} (\partial_x \overline{u})^{\gamma}.$$

Then the equation takes the form

$$\partial_t u = i\partial_x^2 u + u^l u^{-\beta} (\partial_x \overline{u})^{\gamma}.$$

Apply ∂_x^k, $k \geq 3$, and separate terms in the Leibniz expansion of F:

$$\partial_t(\partial_x^k u) = i\partial_x^2(\partial_x^k u) + \sum_{j=0}^{k-1} cj\partial_x^{k-j}(u^{\alpha}u^{-\beta})\partial_x^j(\partial_x\overline{u})^{\gamma} + u^{\alpha}u^{-\beta}\partial_x^k(\partial_x\overline{u})^{\gamma}$$
$$= i\partial_x^2(\partial_x^k u) + \sum c_j \partial_x^{k-1-j}[(\alpha-1)u^{\alpha-1}(\partial_x u)u^{-\beta} + (\beta-1)u^{\alpha}\overline{u}^{\beta-1}(\partial_x\overline{u})]\partial_x^j(\partial_x\overline{u})^{\gamma}$$
$$+u^{\alpha}\overline{u}^{\beta}\partial_x^k(\partial_x\overline{u})^{\gamma}$$
$$= i\partial_x^2(\partial_x^k u) + I + II$$

We further separate term II as:

$$II = \gamma u^{\alpha}\overline{u}^{\beta}(\partial_x\overline{u})^{\gamma-1}\partial_x^{k+1}\overline{u} + u^{\alpha}\overline{u}^{\beta} \sum_{\substack{v\geq 2; j\geq 1,\dots,j_v\geq 1\\ j_1+\dots+j_v=k}} C_{j,v}(\partial_x\overline{u})^{\gamma-\nu}\partial_x^{j_1+1}\overline{u}\cdots\partial_x^{j_v+1}\overline{u}$$
$$= II_1 + II_2$$

Taking together the last relation with $\partial_x^k\overline{u}$, integrating, and taking the real part of term I, we have

$$\sum C_j \mathrm{Re}\int \partial_x^{k-1-j}[(\alpha-1)u^{\alpha-1}(\partial_x u)\overline{u}^{\beta} + (\beta-1)u^{\alpha}\overline{u}^{\beta-1}\partial_x\overline{u}]\partial_x^j(\partial_x\overline{u})^{\gamma}\partial_x^k\overline{u}$$
$$\leq C||\partial_x^{k-1-j}[(\alpha-1)u^{\alpha-1}(\partial_x u)\overline{u}^{\beta} + (\beta-1)u^{\alpha}\overline{u}^{\beta-1}\partial_x\overline{u}]\partial_x^j(\partial_x\overline{u})^{\gamma}||_{L^2}||\partial_x^k\overline{u}||_{L^2}$$
$$\leq C(||u^{\alpha-1}(\partial_x u)\overline{u}^{\beta}||_{H^{k-1}} + ||u^{\alpha}\overline{u}^{\beta-1}\partial_x\overline{u})||_{H^{k-1}})||(\partial_x\overline{u})^{\gamma}||_{H^{k-1}}$$

and use that H^{k-1} is an algebra for term II_1,

$$\mathrm{Re}\int \gamma u^{\alpha}\bar{u}^{\beta}\left(\partial_x\bar{u}\right)^{\gamma-1}\partial_x^{k+1}\bar{u}\partial_x^k\bar{u} \leq \gamma||u^{\alpha}\bar{u}^{\beta}\left(\partial_x\bar{u}\right)^{\gamma-1}||_{L^{\infty}}||\partial_x^{k+1}\bar{u}||_{L^2}||\partial_x^k\bar{u}||_{L^2}$$

Use that $\partial_x^{k+1}\bar{u}\partial_x^k\bar{u} = \frac{1}{2}\partial_x\left(\partial_x^k\bar{u}\right)^2$, and integrate by parts for term II_2,

$$\mathrm{Re}\int \sum_{\substack{v\geq 2\\ j_1\geq 1,\dots j_v\geq 1\\ j_1+\dots+j_v=k}} u^{\alpha}\overline{u}^{\beta}(\partial_x\overline{u})^{\gamma-v}\partial_x^{j_1+1}\overline{u}\dots\partial_x^{j+1}\overline{u}\partial_x^k\overline{u}$$
$$\leq \sum ||u^{\alpha}\overline{u}^{\beta}(\partial_x\overline{u})^{\gamma-v}||_{L^{\infty}}||[\partial_x^{j_1+1}\overline{u}\dots\partial_x^{j_v+1}\overline{u}||_{L^2}||\partial_x^k\overline{u}||_{L^2}$$
$$\leq \sum ||u^{\alpha}\overline{u}^{\beta}(\partial_x\overline{u})^{\gamma-v}||_{L^{\infty}}||[\partial_x^{j_1-1}(\partial_x^2\overline{u})\dots[\partial_x^{j_v-1}(\partial_x^2\overline{u})]||_{L^2}||\partial_x^k\overline{u}||_{L^2}$$

Since $(j_1-1)+\dots+(j_v-1) = k-v \leq k-2,$ use that is in algebra. □

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