Математички Билтен
**28** (LIV)
2004 (81–90)
Скопје, Македонија

ISSN 0351–336X

# SUBCRITICAL HOPF BIFURCATION IN GEORGE SYSTEM

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## Abstract

In this paper we further explore the bifucation and the stability of a dual system to Lorenz system, called George system. The system has a rich structure, possessing a subcritical Hopf bifurcation and a chaotic attractor. The equilibrium point studied is unstable. Using a rigorous mathematical analysis based on symbolic computations, there are obtained some subtle insights on stability and bifurcation.

## Introduction

In the last decades the nonlinear system was intensively studied because the nonlinear phenomena are met in many areas, from engineering to human brain [3, 4] and heart disease. Chaos is a phenomenon close related to nonlinear systems. In [5] it is studied the Duffing oscillator modified and are found the conditions to transition to chaos. It is known that Lorenz equations are derived from the hydrodynamics Navier-Stokes equations. As far as we know, not many systems with an attractor derived from Lorenz system was studied.

The following nonlinear three-dimensional differential system

$$\begin{cases} x' = a(y - x) \\ y' = (c - a)x - axz \\ z' = xy - bz \end{cases} \tag{1}$$

with $a, b, c$ real numberts, $a \neq 0$, is derived from the Lorenz system. We call it George system to distinguish it from Chen system [1] and Lü system [8], which are another two systems derived from Lorenz systems. It help us to understand better the family of Lorenz systems.

## Stability and bifurcation. Results

Bellow can be seen the chaotic attractor of George system for a particular parametric vector.

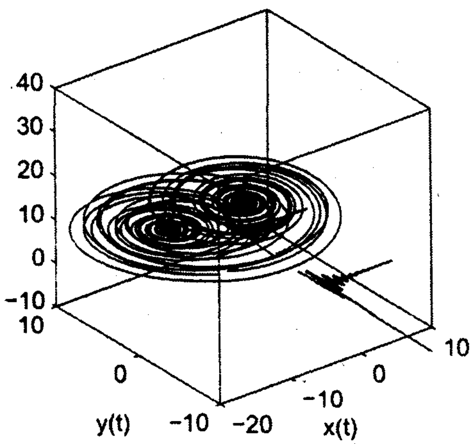


Fig. 1

The attractor of George system. The phase portrait is obtained for parametric vector $(a, b, c) = (2, 3, 20)$

First we recall some results regarding George system.

If $\frac{b}{a}(c-a) > 0$ the system (1) has three isolated equilibria:

$$O(0,0,0),\ S_1\Big(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\Big),$$

$$S_2\Big(-\sqrt{\frac{b}{a}(c-a)}, -\sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\Big)$$

and for $b \neq 0$, $\frac{b}{a}(c-a) \leq 0$ it has only one isolated equilibrium $O(0,0,0)$.

If $b \neq 0$ the following assertions are true:

a) $O(0,0,0)$ is asymptotically stable if and only if

$$(a>0, b>0, 4c-3a\leq 0) \quad \text{or} \quad (a>0, b>0, 4c-3a>0, c\leq a).$$

b) $O(0,0,0)$ is unstable if and only if

$$(b<0 \quad \text{or} \quad (a<0) \quad \text{or} \quad (a>0), 4c-3a>0, c>a.$$

**Remark 1.** The system (1) is conservative if and only if $a+b=0$. Because the system is invariant under the transformation $(x,y,z)\to(-x,-y,z)$, one only needs to consider the stability of system (1) at

$$S_1\left(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\right).$$

Under the linear transformations $(x,y,z)\to(X,Y,Z)$

$$\begin{cases} x = X+\sqrt{\frac{b}{a}(c-a)} \\ y = Y+\sqrt{\frac{b}{a}(c-a)} \\ z = Z+\frac{c-a}{a} \end{cases}$$

system (1) becomes:

$$\begin{cases} X' = a(Y-X) \\ Y' = -a\sqrt{\frac{b}{a}(c-a)}\,Z - aXZ \\ Z' = \sqrt{\frac{b}{a}(c-a)}\,(X+Y) - bZ + XY \end{cases} \tag{2}$$

hence, one has to consider the stability of system (2) at $(0,0,0)$. The Jacobian matrix of system (2) at the point $O(0,0,0)$ is

$$J(E_1) = \begin{pmatrix} a & a & 0 \\ 0 & 0 & -a\sqrt{\frac{b}{a}(c-a)} \\ \sqrt{\frac{b}{a}(c-a)} & \sqrt{\frac{b}{a}(c-a} & -b \end{pmatrix}$$

with characteristic equation

$$\lambda^3 + \lambda^2(a+b) + bc\lambda + 2ab(c-a) = 0. \qquad (*)$$

The equilibrium point

$$S_1\Big(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\Big)$$

is asymptotically stable if and only if

$$(a+b>0, ab(c-a)>0, b(2a^2+bc-ac)>0).$$

The condition of $\frac{b}{a}(c-a)>0$ and the equation $(*)$ having roots with zero real parts is equivalent to $(a,b,c)\in\Omega$, where

$$\Omega = \{(a,b,c)\in \mathbf{R}^3 \mid b>0, a>b, 2a^2+bc=ac\}.$$

In this case the solutions of equation $(*)$ are

$$\lambda_1 = \frac{2a^2-2ac}{c}, \quad \lambda_{2,3} = \pm i\sqrt{ac-2a^2}.$$

Because for $b = b_s := \dfrac{ac-2a^2}{c}$ equation $(*)$ has a negative solution $\lambda_1 = \dfrac{2a^2-2ac}{c} < 0$ together with a pair of purely imaginary roots $\lambda_{2,3} = \pm i\sqrt{ac-2a^2}$ such that $\mathrm{Re}\left(\lambda_b'(b_s)\right) \neq 0$, George system (1) displays a Hopf bufurcation at the point $S_1$.

**Theorem.** *If* $c = \frac{5a}{2}$ *and* $(a,b,c)\in\Omega$ *the point* $S_1\Big(\sqrt{\frac{3}{10}a}, \sqrt{\frac{3}{10}a}, \frac{3}{2}\Big)$ *of George system is unstable and the Hopf bifurcation at the point* $S_1\Big(\sqrt{\frac{3}{10}a}, \sqrt{\frac{3}{10}a}, \frac{3}{2}\Big)$ *is subcritical for any* $a>0$.

**Proof.** In this case the system (2) becomes

$$\begin{cases} X' = a(Y-X) \\ Y' = -XZa - Za\sqrt{\frac{3}{10}a} \\ Z' = XY - \frac{1}{5}Za + (X+Y)\sqrt{\frac{3}{10}a} \end{cases} \qquad (3)$$

and $\lambda_1 = -\frac{6}{5}a$, $\lambda_{2,3} = \pm\frac{1}{2}ia\sqrt{2}$ with $a > 0$ are the eigenvalues of the Jacobian matrix of this system. Then, the eigenvectors corresponding to $\lambda_1 = -\frac{3}{2}a$, $\lambda_2 = ia\frac{\sqrt{2}}{2}$, $\lambda_3 = -ia\frac{\sqrt{2}}{2}$ are, respectively

$$u_1=\begin{pmatrix} -\frac{25}{6}\sqrt{\frac{3}{10}a} \\ \frac{5}{6}\sqrt{\frac{3}{10}a} \\ 1 \end{pmatrix},\quad u_2=\begin{pmatrix} \sqrt{\frac{2a}{15}}\,(1+i\sqrt{2}) \\ i\sqrt{\frac{3}{5}a} \\ 1 \end{pmatrix} \text{ and } u_3=\begin{pmatrix} \sqrt{\frac{2a}{15}}\,(1-i\sqrt{2}) \\ -i\sqrt{\frac{3}{5}a} \\ 1 \end{pmatrix}.$$

Then, the vectors

$$v = \frac{u_2+u_3}{2} = \begin{pmatrix} \sqrt{\frac{2a}{15}} \\ 0 \\ 1 \end{pmatrix},\quad w = \frac{u_2-u_3}{2i} = \begin{pmatrix} 2\sqrt{\frac{a}{15}} \\ \sqrt{\frac{3}{5}a} \\ 0 \end{pmatrix}$$

and $u_1$ lead to the following transformation of the system (3):

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2a}{15}} & 2\sqrt{\frac{a}{15}} & -\frac{25}{6}\sqrt{\frac{3}{10}a} \\ 0 & \sqrt{\frac{3}{5}a} & \frac{5}{6}\sqrt{\frac{3}{10}a} \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} \tag{4}$$

or, equivalently:

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} \frac{6}{97\sqrt{a}}\sqrt{30} & -\frac{4}{97\sqrt{a}}\sqrt{30} & \frac{85}{97} \\ \frac{5}{97\sqrt{a}}\sqrt{15} & \frac{29}{97\sqrt{a}}\sqrt{15} & -\frac{5}{97}\sqrt{2} \\ -\frac{6}{97\sqrt{a}}\sqrt{30} & \frac{4}{97\sqrt{a}}\sqrt{30} & \frac{12}{97} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{5}$$

namely,

$$\begin{cases} X_1 = \frac{6}{97}\frac{X}{\sqrt{a}}\sqrt{30} - \frac{4}{97}\frac{Y}{\sqrt{a}}\sqrt{30} + \frac{85}{97}Z \\ Y_1 = \frac{5}{97}\frac{X}{\sqrt{a}}\sqrt{15} + \frac{29}{27}Y\sqrt{15}\sqrt{\frac{1}{a}} - \frac{5}{97}Z\sqrt{2} \\ Z_1 = -\frac{6}{97}\frac{X}{\sqrt{a}}\sqrt{30} + \frac{4}{97}\frac{Y}{\sqrt{n}}\sqrt{30} + \frac{12}{97}Z. \end{cases} \tag{6}$$

After some calculations, the sistem (3) is transformed into a new form (7) such that center manifold theory can be applied.

$$\begin{cases} X_1' = \frac{1}{2}aY_1\sqrt{2} + \frac{8}{97}aX_1^2 - \frac{167}{582}aX_1Z_1 + \frac{34}{97}aY_1^2 - \\ \qquad - \frac{3325}{2328}aZ_1^2 + \frac{25}{97}aX_1Y_1\sqrt{2} - \frac{1009}{1164}aY_1Z_1\sqrt{2} \\ Y_1' = -\frac{1}{2}aX_1\sqrt{2} + \frac{1817}{1164}aX_1Z_1\sqrt{2} - \frac{283}{582}aY_1Z_1 - \frac{60}{97}aX_1Y_1 - \\ \qquad - \frac{29}{27}aX_1^2\sqrt{2} - \frac{2}{97}aY_1^2\sqrt{2} + \frac{4475}{2328}aZ_1^2\sqrt{2} \\ Z_1' = -\frac{6}{5}aZ_1 + \frac{44}{97}aX_1Z_1 - \frac{8}{97}aX_1^2 + \frac{24}{485}aY_1^2 + \\ \qquad + \frac{75}{194}aZ_1^2 - \frac{28}{485}aX_1Y_1\sqrt{2} - \frac{21}{97}aY_1Z_1\sqrt{2}. \end{cases} \tag{7}$$

Then, the 2-dimensional local center manifold of system (7) near the origin is the set

$$W^c_{loc}(O_1) = \{(X_1, Y_1, Z_1) \in \mathbf{R}^3 \mid Z_1 = h(X_1, Y_1), |X_1| + |Y_1| \ll 1\}$$

where

$$h(0,0) = \frac{\partial h}{\partial X_1}(0,0) = \frac{\partial h}{\partial Y_1}(0,0) = 0.$$

With the substitution $Z_1 = h(X_1, Y_1)$ in (7), the vector field on the center manifold is:

$$\begin{cases} X_1' = \frac{1}{2}aY_1\sqrt{2} + \frac{8}{97}aX_1^2 - \frac{167}{582}aX_1h + \frac{34}{97}aY_1^2 - \\ \qquad - \frac{3325}{2328}ah^2 + \frac{25}{97}aX_1Y_1\sqrt{2} - \frac{1009}{1164}aY_1h\sqrt{2} \\ Y_1' = -\frac{1}{2}aX_1\sqrt{2} + \frac{1817}{1164}aX_1h\sqrt{2} - \frac{283}{582}aY_1h - \frac{60}{97}aX_1Y_1 - \\ \qquad - \frac{29}{97}aX_1^2\sqrt{2} - \frac{2}{97}aY_1^2\sqrt{2} + \frac{4475}{2328}ah^2\sqrt{2}. \end{cases} \tag{8}$$

Assume that $Z_1 = h(X_1, Y_1) = a_{11}X_1^2 + a_{12}X_1Y_1 + a_{22}Y_1^2 + \cdots$.

Substituting $X_1 = w + u$, $Y_1 = i(w - u)$, with $u = \overline{w}$, system (8) leads to:

$$\begin{aligned} w' =& \frac{1}{2} iaw\sqrt{2} + \frac{26}{97} iaw^2\sqrt{2} + \frac{29}{291} ahu - \frac{75}{194} ahw + \\ &+ \frac{42}{97} auw - \frac{3325}{4656} ah^2 + \frac{17}{97} au^2 - \frac{43}{97} aw^2 - \frac{101}{291} iahu\sqrt{2} - \\ &- \frac{471}{388} iahw\sqrt{2} + \frac{31}{97} iauw\sqrt{2} - \frac{4475}{4656} iah^2\sqrt{2} + \frac{1}{97} iau^2\sqrt{2} \end{aligned}$$

$$\begin{aligned} u' =& -\frac{1}{2} iau\sqrt{2} + \frac{29}{291} ahw - \frac{75}{194} ahu + \frac{42}{97} auw - \frac{3325}{4656} ah^2 - \\ &- \frac{43}{97} au^2 + \frac{17}{97} aw^2 + \frac{471}{388} iahu\sqrt{2} + \frac{101}{291} iahw\sqrt{2} - \\ &- \frac{31}{97} iauw\sqrt{2} + \frac{4475}{4656} iah^2\sqrt{2} - \frac{26}{97} iau^2\sqrt{2} - \frac{1}{97} iaw^2\sqrt{2}. \end{aligned}$$

In the new complex variables, $Z_1$ is of the form:

$$Z_1 = N_{11}w^2 + N_{12}wu + N_{22}u^2 + O(|w|^3) \tag{9}$$

with

$$Z_1' = 2N_{11}w'w + N_{12}(w'u + wu') + 2N_{22}u'u + O(|w|^3)$$

and cusing the above forms of $w'$ and $u'$, we hawe:

$$Z_1' = iaw^2 N_{11}\sqrt{2} - iau^2 N_{22}\sqrt{2} + O(|w|^3). \tag{10}$$

On the other hand, from (7) and (9) we have:

$$\begin{aligned} Z_1' =& \frac{6}{5} a\Big(u^2\Big(\frac{14}{291} i\sqrt{2} - \frac{32}{291}\Big) - uwN_{12} - u^2N_{22} - w^2N_{11} - \frac{16}{291} uw\Big) + \\ &+ \frac{6}{5} aw^2\Big(-\frac{14}{291} i\sqrt{2} - \frac{32}{291}\Big) + O(|w|^3). \end{aligned} \tag{**}$$

Identifying coefficiennts of $w^2, wu, u^2$ in (10) and (**) one can find:

$$N_{11} = \frac{76}{4171} i\sqrt{2} - \frac{332}{4171}, \quad N_{12} = -\frac{16}{291}, \quad N_{22} = -\frac{76}{4171} i\sqrt{2} - \frac{332}{4171}$$

and

$$h = w^2\Big(\frac{76}{4171} i\sqrt{2} - \frac{332}{4171}\Big) - \frac{16}{291} uw - u^2\Big(\frac{76}{4171} i\sqrt{2} + \frac{332}{4171}\Big).$$

Hence, the vector field restricted to the center manifold is:

$$w' = \frac{1}{2}iaw\sqrt{2} + \Big(\frac{26}{97}i\sqrt{2} - \frac{43}{97}\Big)w^2a + \Big(\frac{42}{97} + \frac{31}{97}i\sqrt{2}\Big)auw+ \\ + \Big(\frac{17}{97} + \frac{1}{97}i\sqrt{2}\Big)au^2 + \Big(\frac{9.6}{100}i\sqrt{2} + \frac{2.5}{100}\Big)w^2ua + \cdots .$$

Note by

$$g_{20} = 2\Big(\frac{26}{97}i\sqrt{2} - \frac{43}{97}\Big)a, \quad g_{11} = \Big(\frac{42}{97} + \frac{31}{97}i\sqrt{2}\Big)a,$$

$$g_{02} = 2\Big(\frac{17}{97} + \frac{1}{97}i\sqrt{2}\Big)a, \quad g_{21} = 2\Big(\frac{9.6}{100}i\sqrt{2} + \frac{2.5}{100}\Big)a .$$

The first Lyapunov coefficient is defined as:

$$l_1(0) = \frac{\mathrm{Re}(C(0))}{a}$$

where

$$C(0) = \frac{i}{2a}\Big(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\Big) + \frac{g_{21}}{2} \simeq (6.14 \cdot 10^{-2} - 0.63i)a .$$

Now it is clear that $l_1(0) > (0)$ for any $a > 0$. Consequently, the Hopf bifurcation at the point $S_1$ is subcritical and the point $O(0,0,0)$ of system (3) is unstable, so the point $S_1\Big(\sqrt{\frac{3}{10}a}, \sqrt{\frac{3}{10}a}\,\frac{3}{2}\Big)$ of system (1) is unstable.

## Conclusions

In this paper we further studied a relative new system with a rich srtucture. Are obtained some insights on stability and bifurcation based on some precise symbolic computation. The Hopf bifurcation is subcritical and equilibrium point studied is unstable. These results was obtained using center manifold theory. Surely, there is still a lot of work, and this paper is a step in analyzing this system. Up todey, the bifurcation on general case was not considered because the coefficients of the system on the central manifold are huge.

## Acknowledgements

This work was partially supported through a European Community Marie Curie Fellowship and in the framework of the CTS, contract number HPMT-CT2001-00278. The author is grateful to the host institution, Imperial College of London.

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# ПОДКРИТИЧНИ ХОПФ БИФУРАКЦИИ ВО СИСТЕМ НА GEORGE

Gheorghe Tigan

**Р е з и м е**

Во овој труд ќе извршиме понатамошни испитувања на бифуракцијата и на стабилноста на дуалниот систем на Лоренцовиот, наречен на George систем. Системот има богата структура, поседува подкритична Хопфова бифуракција и хаотичен атрактор. Проучуваната рамнотежна точка е нестабилна. Употребувајќи строга математичка анализа заснована на симболички пресметки, добиени се подобри погледи на стабилноста на бифуракциите.

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