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THE STUDY OF SOME CHARACTERISTICS OF THE SOLUTIONS OF THE ABEL EQUATION OF THE SECOND KIND

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Abstract

In this paper, we study the Abel differential equation of the second kind

$$y' = \frac{f_2(x)y^2+f_1(x)y+f_0(x)}{y+g(x)}.$$

Treating different assumptions for the functions $f_2(x)$, $f_1(x)$, $f_0(x)$ and $g(x)$, we prove five theorems dealing with the qualitative features of the solutions of the above mentioned equation.

It is known that the Abel equation of the second kind

$$y' = \frac{f_2(x)y^2 + f_1(x)y + f_0(x)}{y + g(x)}. \tag{1}$$

has been studied a lot from the point of obtaining the solution under some assumptions for the functions $f_2(x)$, $f_1(x)$, $f_0(x)$ and $g(x)$ ([1]). The exploring of the qualitative properties of the solutions is found, for example, in [2] and [4]. In this paper we obtain some qualitative properties of the solutions of the differential equation (1), under certain assumptions for the above mentioned functions contained in the equation.

Let us consider the equation (1) and let the functions $f_2(x)$, $f_1(x)$, $f_0(x)$ and $g(x)$ be continuous. Further, suppose that the conditions:

$$f_0(x) < 0,\ f_2(x) > 0,\ g(x) > 0,\ y > -g(x) \quad \text{for} \quad x \geq x_0\,; \tag{2}$$

$$\lim_{x\to+\infty} \frac{f_0(x)}{f_2(x)} = A \quad \text{and} \quad \lim_{x\to+\infty} \frac{f_1(x)}{f_2(x)} = B\,, \tag{3}$$

are fulfilled, where A and B are real constants. Then we have:

1. Function $D(x) = f_1^2(x) - 4f_0(x)\,f_2(x)$ is positive for $x \ge x_0$, and therefore, there exist real functions $y_1(x)$ and $y_2(x)$, such ones that:

$$y' = \frac{f_2(x)(y - y_1)(y - y_2)}{y + g(x)}\,. \tag{4}$$

The curves $y_1(x)$ and $y_2(x)$ are the curves of stationary points for the solutions $y(x)$ of the equation (1). From the relations $y_2(x) - y_1(x) = \frac{\sqrt{D(x)}}{f_2(x)}$ and $y_1(x)\,y_2(x) = \frac{f_0(x)}{f_2(x)} < 0$, it follows that $y_1(x) < 0$ and $y_2(x) > 0$, for $x \ge x_0$. The analytic expressions for these functions are:

$$\begin{aligned} y_1(x) &= -\frac{1}{2}\left(\sqrt{\left(\frac{f_1(x)}{f_2(x)}\right)^2 - 4\,\frac{f_0(x)}{f_2(x)}} + \frac{f_1(x)}{f_2(x)}\right), \\ y_2(x) &= \frac{1}{2}\left(\sqrt{\left(\frac{f_1(x)}{f_2(x)}\right)^2 - 4\,\frac{f_0(x)}{f_2(x)}} - \frac{f_1(x)}{f_2(x)}\right). \end{aligned} \tag{5}$$

Taking the limit values in relation (5), we obtain:

$$\lim_{x\to+\infty} y_1(x) = \frac{1}{2}\left(\sqrt{B^2 - 4A} + B\right) = c_1\,,$$

$$\lim_{x\to+\infty} y_2(x) = \frac{1}{2}\left(\sqrt{B^2 - 4A} - B\right) = c_2\,.$$

Therefore, functions $y_1(x)$ and $y_2(x)$ are continuous and have finite limit values and, hence, they are bounded.

2. The right side of the equation (1) is continuous in the region $\Omega = \{(x, y): x \ge x_0,\, y > -g(x)\}$ and the partial derivative with respect to y of the right side is continuous on any closed and bounded subset Ω. Hence, the conditions for the existence and uniqueness of the solutions in Ω are fulfilled.

3. In the region $\Omega_+ = \{(x, y) \in \Omega: y < y_1(x) \vee y > y_2(x)\}$ is $y' > 0$, until in the domain $\Omega_- = \{(x, y) \in \Omega: y_1(x) < y < y_2(x)\}$ is $y' < 0$.

First, let us consider the qualitatiive properties of the solutions of the equation (1), for the case when $y_1(x)$ and $y_2(x)$ are monotonous functions. Moreover, together with some additional conditions, we are going to suppose that conditions (2), (3) and

$$\sup_{x\ge x_0}\,\bigl(-g(x)\bigr) < \inf_{x\ge x_0}\; y_1(x) \tag{6}$$

are satisfied. Hence, functions $y_1(x)$ and $y_2(x)$ are monotonous increasing functions (the similar reasoning could be deduced in other cases, also) and let

$$A = \lim_{x \to \infty} \frac{f_0(x)}{f_2(x)} \neq 0 . \tag{7}$$

Then we have the following theorems:

Theorem 1. *There exists at least one bounded positive solution of the equation* (1) *with the property* $\lim_{x \to +\infty} y(x) = c_2$.

Proof: First, it is easy to see that, under the above assumptions, $c_1 < 0$ $c_2 > 0$. On the straight line $y = c_2$ is $y' > 0$. On the other hand, the points on the curve $y_2(x)$ are the maximum of the solutions, since $y' > 0$ for $y > y_2(x)$, and $y' < 0$ for $y < y_2(x)$ and finally, $y' = 0$ for $y = y_2(x)$. The straight line $y = c_2$ and the curve $y = y_2(x)$ can be taken as, respectively, the top, or, the bottom margin of the retract–tube. Using the retract method ([2]), we conclude that there exists at least one solution $y(x)$ in the strip $y_2(x) < y < c_2$, $(x \geq x_0)$. The solution tends towards c_2, since $\lim_{x \to +\infty} y_2(x) = c_2$.

Theorem 2. *There exist infinitely many bounded solutions of the equation* (1), *from which a class of negative solutions can be easily selected.*

Proof: Let us observe the solutions starting from the curve $y = y_2(x)$ or from the region Ω. They are monotonously decreasing and, if they don't reach $y_1(x)$, they remain between the bounded functions $y_1(x)$ and $y_2(x)$. If, contrary, any of these solutions reaches $y_1(x)$ for some $x = x^*$, then, at point x^* that solution has a minimum and for $x > x^*$ the solution monotonously increases, so that $y(x) < y_1(x)$ for $x > x^*$.

Indeed, if for some $x_1 > x^*$ it would be $y(x_1) = y_1(x_1)$, then the point $x = x_1$ would be the zero of the function $\varphi(x) = y(x) - y_1(x)$, where $\varphi'(x_1) = y'(x_1) - y_1'(x_1) = -y_1'(x_1) < 0$. Since the function $\varphi(x)$ is continuous, there exists $\delta > 0$ such that $\varphi(x) > 0$, i.e., $y(x) > y_1(x)$ for $x \in (x_1 - \delta, x_1)$, and that is a contradiction to the assumption that $y(x) < y_1(x)$ for $x \in (x^*, x_1)$.

From this class of the solutions, we can extract the solutions starting from the strip $y_1(x) < y < 0$, and which are evidently negative. The solutions starting from the strip $\sup_{x \geq x_0} \big(-g(x)\big) < y < y_1(x)$ also belong to this class.

Exmple 1. Let the given differential equation be

$$y' = \frac{(3x+1)y^2 - \frac{3}{4}(x-1)y - \frac{(3x+2)(9x+1)}{8(3x+1)}}{y + \sqrt{x} + 1.5}.$$

Here $y_1(x) = -\frac{3x+2}{2(3x+1)}$ and $y_2(x) = \frac{9x+1}{4(3x+1)}$ are monotonously increasing functions for $x \geq x_1$, where $A = -0.375$, $B = -0.25$, $c_1 = -0.5$ and $c_2 = 0.75$. The graph of the solutions obtained by using a computer is given in Figure 1.

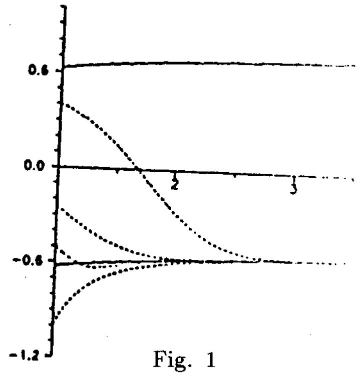

Fig. 1

In the following theorem we establish the strip stable for the solutions of the equation (see [3]).

Theorem 3. *The strip*

$$A = \{(x, y): y_1(x_0) < y < c_1,\, x \geq x_0\} \tag{8}$$

is stable for the solutions of the equation (1).

Proof: The curve of the stationary points $y_1(x)$ obviously belongs to the strip (8) for $x \geq x_0$. Let $\varepsilon > 0$ and choose

$\delta < \min\{\varepsilon, -c_1, y_1(x_0) - \sup\limits_{x \geq x_0}(-g(x))\}$. Consequently, $c_1 + \varepsilon < 0$ and

$y_1(x_0) - \varepsilon > \sup\limits_{x \geq x_0}(-g(x))\}$. Using the above noticed properties of the solutions and the monotony of the function $y_1(x)$, the following cases are possible:

1. The solutions with the initial condition $y_1(x_0) < y_0 \leq c_1$ are monotonously decreasing, they reach the minimum on the curve $y_1(x)$ and afterwards they monotonously increase, being under $y_1(x)$ all the time.

2. For $c_1 < y_0 < c_1 + \varepsilon$ the solutions monotonously decrease (above the straight line $y = c_1$), or they have the property as solutions in the case 1.

3. If $y_1(x_0) - \varepsilon < y_0 \leq y_1(x_0)$, the solutions monotonously increase and they are under the curve $y_1(x)$ for $x > x_0$.

In all the three considered cases, the solution is between the straight lines $y_1 = y_1(x_0) - \varepsilon$ and $y = c_1 + \varepsilon$. This proves the stability of the strip A for the solutions of the equation (1).

Now, let us consider the differential equation (1) under the conditions (2), (3), (6) and (7), but without the assumption of the monotony of the curves of stationary points $y_1(x)$ and $y_2(x)$. Then we have the following two theorems:

Theorem 4. *The equation* (1) *has at least one bounded positive solution,* *Proof.* Since the function $y_2(x)$ is bounded and continuous, and using the fact that $\lim\limits_{x\to\infty} y_2(x) = c_2 > 0$, it follows that there exist constants a and b such that $0 < a < b$ and $a < \inf\limits_{x\geq x_0} y_2(x)$, $b > \sup\limits_{x\geq x_0} y_2(x)$. The solutions have the positive (negative) coefficient of the direction on the straight line $y = a (y = b)$, respectively. These straight lines can be taken as the margins of the retract–tube in the retract method, which confirms the existence of at least one solution located between the lines $y = a$ and $y = b$.

Theorem 5. *The equation* (1) *has a class of the bounded negative solutions.*

Proof. The solutions starting from the strip $y_2(x) < y < 0$ are monotonously decreasing and, if they don't reach $y_2(x)$, they are tending towards a negative constant. On the contrary, they are linked to $y_1(x)$, reaching the minimum on it.

The similar conclusion can be obtained for the solutions starting from the strip $\sup\limits_{x\geq x_0} \big(-g(x)\big)\ y < y_1(x)$.

Example 2. Let the given differential equation be

$$y' = \frac{y^2 + x(e^{-x} - e^{-x^2})y - (xe^{-x} + 0.125)(xe^{-x^2} + 0.5)}{y+1}.$$

In this example the curves of the stationary points $y_1(x) = -xe^{-x^2} - 0.5$ and $y_2(x) = xe^{-x} + 0,125$ are not monotonous functions for $x \geq x_0$. Several solutions of the equation are given in Figure 2.

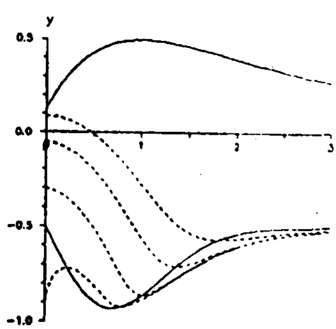

Fig. 2

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ПРОУЧУВАЊЕ НА НЕКОИ ОСОБИНИ НА РЕШЕНИЈАТА НА АБЕЛОВАТА РАВЕНКА ОД ВТОР РЕД

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Р е з и м е

Во оваа работа е проучувана Абеловата диференцијална равенка (1). При различни претпоставки за функциите $f_2(x)$, $f_1(x)$, $f_0(x)$ и $g(x)$ се докажуваат пет теореми за квалитативни особини на решенијата на погоре дадената равенка.

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