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STRONG n–CONVEX n–NORMED SPACES

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Abstract

The concept of a n–skalar product on a vector space of dimension greater than $n-1$ and n–norm, introduced by A. Misiak ([6]), is a multidimensional analog of the scalar product and the norm. In [6] and [5] are proved the basic properties of a n–preHilbert and n–normed space. In this work we will give a generalisation and some properties of a strong 2–convex 2–normed spaces which was treat in [2] and [3].

Let n be a natural member, L a real vector space such that dim $L \geq n$ and $(\bullet, \bullet \mid \bullet, \dots, \bullet)$ be a real function on L^{n+1} such that

i) $(a, a \mid x_1, \dots, x_{n-1}) \geq 0$, for each $a, x_1, \dots, x_{n-1} \in L$ and $(a, a \mid x_1, \dots, x_{n-1}) = 0$ if and only if $a, x_1, \dots, x_{n-1}$ are lineary dependent;

ii) $(a, b \mid x_1, \dots, x_{n-1}) = \big(\varphi(a), \varphi(b) \mid \pi(x_1), \dots, \pi(x_{n-1})\big)$, for each $a, b, x_1, \dots, x_{n-1} \in L$ and for every bejections $\pi: \{x_1, \dots, x_{n-1}\} \to \{x_1, \dots, x_{n-1}\}$ and $\varphi: \{a, b\} \to \{a, b\}$;

iii) $(a, a \mid x_1, x_2 \dots, x_{n-1}) = (x_1, x_1 \mid a, x_2 \dots, x_{n-1})$, for every $a, x_1, \dots, x_{n-1} \in L$;

iv) $(\alpha a, b \mid x_1, \dots, x_{n-1}) = \alpha(a, b \mid x_1, \dots, x_{n-1})$, for every $a, b, x_1, \dots, x_{n-1} \in L$ and for every $\alpha \in R$; and

v) $(a+a_1, b \mid x_1, \dots, x_{n-1}) = (a, b \mid x_1, \dots, x_{n-1}) + (a_1, b \mid x_1, \dots, x_{n-1})$, for every $a, b, a_1, x_1, \dots, x_{n-1} \in L$.

We call function $(\bullet, \bullet \mid \bullet, \dots, \bullet)$ n–scalar product and we call $\big(L, (\bullet, \bullet \mid \bullet, \dots, \bullet)\big)$ n–preHilbert space.

Let L be a real vector space of dimension greater of $n-1$ and $\|\bullet, \ldots, \bullet\|$ is a real function on L^n with the following conditions:

i) $\|x_1, \ldots, x_n\| \geq 0$, $\forall\, x_1, \ldots x_n \in L$, and $\|x_1, \ldots, x_n\| = 0$ if and only if the set $\{x_1, \ldots, x_n\}$ is lineary depend;

ii) $\|x_1, \ldots, x_n\| = \|\pi(x_1), \ldots, \pi(x_n)\|$, for every $x_1, \ldots, x_n$ and every bejection $\pi\colon \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}$;

iii) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \cdot \|x_1, \ldots, x_n\|$, for every $x_1, \ldots, x_n \in L$ and every $\alpha \in R$,

iv) $\|x_1+x_1', \ldots, x_n\| \leq \|x_1, \ldots, x_n\| + \|x_1', \ldots, x_n\|$, for every $x_1, \ldots, x_n$, $x_1' \in L$.

We call the function $\|\bullet, \ldots, \bullet\|$ a n–norm of L, and we call $(L, \|\bullet, \ldots, \bullet\|)$ a n–normed space.

1. n–vectors

In [6] are introduced the concept of a n–vector and are proved some properties of n–vectors. Let n be natural number and L be a real vector space of dimension greater or equal to n. We denote by N_L' the family of all formal notations:

$$\sum_{i=1}^{k} x_1^{(i)} \times x_2^{(i)} \times x_3^{(i)} \times \ldots \times x_n^{(i)}, \quad x_j^{(i)} \in L,\ i=1, 2, \ldots, k;\ j=1, 2, \ldots, n.$$

We define equivalence relation α on N_L' with:

$$\sum_{i=1}^{k} x_1^{(i)} \times x_2^{(i)} \times x_3^{(i)} \times \ldots \times x_n^{(i)} \sim \sum_{i=1}^{t} y_1^{(i)} \times y_2^{(i)} \times y_3^{(i)} \times \ldots \times y_n^{(i)}$$

if and only if for every linear functionals $f_1, f_2, \ldots, f_n$ on L it is true

$$\sum_{i=1}^{k} \begin{vmatrix} f_1(x_1^{(i)}) & f_1(x_2^{(i)}) & \cdots & f_1(x_n^{(i)}) \\ f_2(x_1^{(i)}) & f_2(x_2^{(i)}) & \cdots & f_2(x_n^{(i)}) \\ \cdots & \cdots & \cdots & \cdots \\ f_n(x_1^{(i)}) & f_n(x_2^{(i)}) & \cdots & f_n(x_n^{(i)}) \end{vmatrix} = \sum_{i=1}^{t} \begin{vmatrix} f_1(y_1^{(i)}) & f_1(y_2^{(i)}) & \cdots & f_1(y_n^{(i)}) \\ f_2(y_1^{(i)}) & f_2(y_2^{(i)}) & \cdots & f_2(y_n^{(i)}) \\ \cdots & \cdots & \cdots & \cdots \\ f_n(y_1^{(i)}) & f_n(y_2^{(i)}) & \cdots & f_n(y_n^{(i)}) \end{vmatrix}.$$

We denote with N_L the factor space $N_L'/\sim$. The elements of N_L are called *n–vectors* on L([6]). The elements of N_L' which belongs to one n–vector a called *represents of the* n-vector. The n–vector whose represent is

$$\sum_{i=1}^{k} x_1^{(i)} \times x_2^{(i)} \times x_3^{(i)} \times \ldots \times x_n^{(i)}$$

we denote with

$$N\left(\sum_{i=1}^{k} x_1^{(i)} \times x_2^{(i)} \times x_3^{(i)} \times \ldots \times x_n^{(i)}\right).$$

We say that the n–vector is *simple* if it has a represent of a from $x_1 \times \ldots \times x_n$. The theorem 10 in [6] implies: if $\dim L \le n+1$, than each n–vector is simple. The space N_L of n–vectors on L is real vector space with the operations defined by

$$\alpha N\left(\sum_{i=1}^{k} x_1^{(i)} \times x_2^{(i)} \times x_3^{(i)} \times \ldots \times x_n^{(i)}\right) = N\left(\sum_{i=1}^{k} \alpha x_1^{(i)} \times x_2^{(i)} \times x_3^{(i)} \times \ldots \times x_n^{(i)}\right)$$

and

$$N\left(\sum_{i=1}^{k} x_1^{(i)} \times x_2^{(i)} \times \ldots \times x_n^{(i)}\right) + N\left(\sum_{i=1}^{m} x_1^{(i+k)} \times x_2^{(i+k)} \times \ldots \times x_n^{(i+k)}\right) =$$

$$= N\left(\sum_{i=1}^{k+m} x_1^{(i)} \times x_2^{(i)} \times \ldots \times x_n^{(i)}\right).$$

In N_L the null vector represents has a form $x_1 \times x_2 \times x_3 \times \ldots \times x_n$ where $x_1, x_2, \ldots, x_n \in L$ are lineary depend vectors. Hence, the null vector on the space N_L is simple.

If $\| \bullet \|$ is a norm on N_L, then

$$\|x_1, \ldots, x_n\| = \|N(x_1 \times \ldots \times x_n)\|$$

define a n–norm $\|\bullet, \ldots, \bullet\|$ on L ([6] theorem 15). If $n = 2$, there is an example in [7], p. 52, which show that for every n–norm $\|\bullet, \ldots, \bullet\|$ on L, there is no a norm $\| \bullet \|$ on N_L which satisfy

$$\|x_1, \ldots, x_n\| = \|N(x_1 \times \ldots \times x_n)\|.$$

If all n–vectors are simple, e.t. if $\dim L \le n+1$, then for every n–norm on L there is a norm on N_L which satisfy $\|N(x_1 \times \ldots \times x_n)\| = \|x_1, \ldots, x_n\|$, for every $x_1, \ldots, x_n \in L$. ([6] theorem 19). If $(\bullet, \bullet)$ is a scalar product on N_L, then $(a, b \mid x_1, \ldots, x_{n-1}) = \big(N(a, x_1, \ldots, x_{n-1}), N(b, x_1, \ldots, x_{n-1})\big)$ is a n–scalar product $(\bullet, \bullet \mid \bullet, \ldots, \bullet)$ on L ([6] theorem 11). If $\dim L \le n+1$, then on every n–scalar product $(\bullet, \bullet \mid, \ldots, \bullet)$ on L correspond a scalar product on N_L defined with

$$\big(N(a \times x_1 \times \ldots \times x_{n-1}), N(b \times x_1 \times \ldots \times x_{n-1})\big) = (a, b \mid x_1, \ldots, x_{n-1}),$$

for every $a, b, x_1, \ldots, x_{n-1} \in L$ ([6] theorem 14).

2. n–linear mappings

Definition 1. Let X_i, $i = 1, 2, \ldots, n$ and Y are real vectors spaces. We call a n–linear operator $A: X_1 \times X_2 \times \ldots \times X_n \to Y$ every function $A(x_1, \ldots, x_n)$, $x_i \in X_i$, $i = 1, \ldots, n$ which is linear in each own argument. If Y is a set of real numbers, then n–linear operator we call a n–linear functional.

It is easy to see that the operator (functional) A is a n–linear if and only if

i) $A(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = \sum_{z_i \in \{x_i, y_i\},\ i=1,\ldots,n} A(z_1, z_2, \ldots, z_n)$,

and

ii) $A(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_1 \alpha_2 \ldots \alpha_n A(x_1, x_2, \ldots, x_n)$, for every $\alpha_i \in R$, $i = 1, 2, \ldots, n$.

Let X be a vector space. We denote with $T_{L,X}$ the family of all mappings $T: L^n \to X$ such that for every T there is a linear mapping $f: N_L \to X$ which satisfy

$$f\big(N(x_1 \times \ldots \times x_n)\big) = T(x_1, \ldots, x_n), \quad \text{for every} \quad x_1, \ldots, x_n \in L .$$

It is clear that the mapping f is well defined.

Theorem 1. $T \in T_{L,X}$ if and only if T is n-linear and $T(x_1, \ldots, x_n) = 0$, for every linear dependent vectors $x_1, \ldots, x_n \in L$.

Proof. Let $T \in T_{L,X}$. Since f is linear and the space N_L has the properties $N(x_1 \times \ldots \times x_n) = \pm N\big(\pi(x_1) \times \ldots \times \pi(x_n)\big)$, for every bejection $\pi: \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}$ and

$$\alpha N(x_1 \times x_2 \times \ldots \times x_n) + \beta N(y_1 \times x_2 \times \ldots \times x_n) = N\big((\alpha x_1 + \beta y_1) \times x_2 \times \ldots \times x_n\big),$$

we conclude that T is n–linear. But, since for the linear dependent vectors $x_1, \ldots, x_n \in L$ it is true $N(x_1 \times \ldots \times x_n) = 0$, we get

$$T(x_1, \ldots, x_n) = f\big(N(x_1 \times \ldots \times x_n)\big) = f(0) = 0 .$$

Conversly, let T ia a n–linear and for every linear dependent vectors $x_1, \ldots, x_n \in L$ it is true that $T(x_1, \ldots, x_n) = 0$. If

$$N = N\left(\sum_{i=1}^{m} x_1^{(i)} \times \ldots \times x_n^{(i)}\right) \in N_L ,$$

we define

$$f(N) = \sum_{i=1}^{m} T\left(x_1^{(i)}, \ldots, x_n^{(i)}\right) .$$

Using the caracterisation of the n–vectors, given in [6], it is easy to see that $f(N)$ is independent of the choice of the represent of N. If

$$N_1 = N\left(\sum_{i=1}^{m} x_1^{(i)} \times \ldots \times x_n^{(i)}\right), N_2 = N\left(\sum_{j=1}^{k} y_1^{(j)} \times \ldots \times y_n^{(j)}\right) \text{ and } \alpha, \beta \in R$$

we have

$$f(\alpha N_1 + \beta N_2) = f\left(N\left(\sum_{i=1}^{m} \alpha x_1^{(i)} \times \ldots \times x_n^{(i)} + \sum_{j=1}^{k} \beta y_1^{(j)} \times \ldots \times y_n^{(j)}\right)\right) =$$

$$= \sum_{i=1}^{m} T\left(\alpha x_1^{(i)}, \ldots, x_n^{(i)}\right) + \sum_{j=1}^{k} T\left(\beta y_1^{(j)}, \ldots, y_n^{(j)}\right) =$$

$$= \alpha \sum_{i=1}^{m} T\left(x_1^{(i)}, \ldots, x_n^{(i)}\right) + \beta \sum_{j=1}^{k} T\left(y_1^{(j)}, \ldots\ y_n^{(j)}\right) =$$

$$= \alpha f(N_1) + \beta f(N_2),$$

which means that f is linear. It means $T \in T_{\mathrm{L,X}}$.

Theorem 2. Let $T \in T_{\mathrm{L,X}}$ and $T(x_1, \ldots, x_n) = 0$ for every linear independent vectors $x_1, \ldots, x_n \in L$.

a) If $\| \bullet \|$ is norm on X, then $\|x_1, \ldots, x_n\| = \|T(x_1, \ldots, x_n)\|$ define a n–norm on L.

b) If $(\bullet, \bullet)$ is a scalar product on X then

$$(a, b \mid x_1, \ldots, x_{n-1}) = \big(T(a, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big)$$

is a n–scalar product on L.

Proof. Let $T \in T_{\mathrm{L,X}}$, $T(x_1, \ldots, x_n) \neq 0$ for every linear independent vectors $x_1, \ldots, x_n \in L$.

a) If $\| \bullet \|$ is a norm on X, then

i) $\|x_1, \ldots, x_n\| = \|T(x_1, \ldots, x_n)\| \geq 0$ for every $x_1, \ldots, x_n \in L$ and $\|x_1, \ldots, x_n\| = 0$ if and only if $\|T(x_1, \ldots, x_n)\| = 0$ if and only if $x_1, \ldots, x_n$ are lineary dependent vectors.

ii) For every bejection $\pi: \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}$ it is true

$$\|x_1, \ldots, x_n\| = \|T(x_1, \ldots, x_n)\| = \|f\big(N(x_1 \times \ldots \times x_n)\big)\| =$$

$$= \|f\big(\pm N\big(\pi(x_1) \times \ldots, \times \pi(x_n)\big)\big)\| =$$

$$= \|f\big(N\big(\pi(x_1) \times \ldots, \times \pi(x_n)\big)\big)\| =$$

$$= \|T\big(\pi(x_1), \ldots, \pi(x_n)\big)\| = \|\pi(x_1), \ldots, \pi(x_n)\|.$$

iii) For every $x_1, \ldots, x_n \in L$ and each $\alpha \in R$ it is true

$$\|\alpha x_1, \ldots, x_n\| = \|T(\alpha x_1, \ldots, x_n)\| = \|\alpha T(x_1, \ldots, x_n)\| = $$
$$= |\alpha| \cdot \|T(x_1, \ldots, x_n)\| = |\alpha| \cdot \|x_1, \ldots, x_n\| .$$

iv) For every $x_1', x_1, \ldots, x_n \in L$ it is true

$$\|x_1 + x_1', x_2 \ldots, x_n\| = \|T(x_1 + x_1', x_2 \ldots, x_n\| =$$
$$= \|T(x_1, \ldots, x_n) + T(x_1', \ldots, x_n)\| \leq$$
$$\leq \|T(x_1, \ldots, x_n)\| + \|T(x_1', \ldots, x_n)\| =$$
$$= \|x_1, \ldots, x_n\| + \|x_1', \ldots, x_n\| .$$

b) If $(\bullet, \bullet)$ is a scalar product on X, then
i) For every $a, x_1, \ldots, x_{n-1} \in L$ it is true

$$(a, a \mid x_1, \ldots, x_{n-1}) = \big(T(a, x_1, \ldots, x_{n-1}),\ T(a, x_1, \ldots, x_{n-1})\big) \geq 0$$

and $(a, a \mid x_1, \ldots, x_{n-1}) = 0$ if and only if $T(a, x_1, \ldots, x_{n-1}) = 0$ e.t. if and only $a, x_1, \ldots, x_{n-1}$ are lineary dependendent vectors.

ii) For every bejections $\pi: \{x_1, \ldots, x_{n-1}\} \to \{x_1, \ldots, x_{n-1}\}$ and $\varphi: \{a, b\} \to \{a, b\}$ it is true

$$(a,b \mid x_1, \ldots, x_{n-1}) = \big(T(a, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) =$$
$$= \big(T\big(\varphi(a), x_1, \ldots, x_{n-1}\big),\ T\big(\varphi(b), x_1, \ldots, x_{n-1}\big)\big) =$$
$$= \big(f\big(N\big(\varphi(a) \times x_1 \times \ldots \times x_{n-1}\big)\big),\ f\big(N\big(\varphi(b) \times x_1 \times \ldots \times x_{n-1}\big)\big)\big) =$$
$$= \big(f\big(\pm N\big(\varphi(a) \times \pi(x_1) \times \ldots \times \pi(x_{n-1})\big)\big), f\big(\pm N\big(\varphi(b) \times \pi(x_1) \times \ldots \times \pi(x_{n-1})\big)\big)\big) =$$
$$= \big(T\big(\varphi(a), \pi(x_1), \ldots, \pi(x_{n-1})\big),\ T\big(\varphi(b)\, \pi(x_1), \ldots, \pi(x_{n-1})\big)\big) =$$
$$= \big(\varphi(a), \varphi(b) \mid \pi(x_1), \ldots, \pi(x_{n-1})\big) .$$

iii) For every $a, x_1, \ldots, x_{n-1} \in L$ it is true

$$(a, a \mid x_1, \ldots, x_{n-1}) = \big(T(a, x_1, \ldots, x_{n-1}),\ T(a, x_1, \ldots, x_{n-1})\big) =$$
$$= \big(f\big(N(a \times x_1 \times \ldots \times x_{n-1})\big),\ f\big(N(a \times x_1 \times \ldots \times x_{n-1})\big)\big) =$$
$$= \big(f\big(N(x_1 \times a \times \ldots \times x_{n-1})\big),\ f\big(N(x_1 \times a \times \ldots \times x_{n-1})\big)\big) =$$
$$= \big(T(x_1, a, \ldots, x_{n-1}),\ T(x_1, a, \ldots x_{n-1})\big) =$$
$$= (x_1, x_1 \mid a, \ldots, x_{n-1}) .$$

iv) For every $a, b, x_1, \ldots, x_{n-1} \in L$ and each $\alpha \in R$ it is true

$$
\begin{aligned}
(\alpha a, b \mid x_1, \ldots, x_{n-1}) &= \big(T(\alpha a, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) = \\
&= \big(\alpha T(a, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) = \\
&= \alpha\big(T(a, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) = \\
&= \alpha(a, b \mid x_1, \ldots, x_{n-1}).
\end{aligned}
$$

v) For every $a, a_1, b, x_1, \ldots, x_{n-1} \in L$ it is true

$$
\begin{aligned}
&(a + a_1, b \mid x_1, \ldots, x_{n-1}) = \\
&= \big(T(a + a_1, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) = \\
&= \big(T(a, x_1, \ldots, x_{n-1}) + T(a_1, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) = \\
&= \big(T(a, x_1, \ldots, x_{n-1},\ T(b, x_1, \ldots, x_{n-1})\big) + \\
&\quad + \big(T(a_1, x_1, \ldots, x_{n-1}),\ T(b, x_1, \ldots, x_{n-1})\big) = \\
&= (a, b \mid x_1, \ldots, x_{n-1}) + (a_1, b \mid x_1, \ldots, x_{n-1}).
\end{aligned}
$$

Definition 2. Let $(L, \|\bullet, \ldots, \bullet\|)$ be a n–normed space. We say that the n–functional f with domain $D(f) \subset L^n$ is bounded, if there exist a real constant $k \geq 0$ such that

$$|f|(x_1, x_2, \ldots, x_n)| \leq k\|x_1, x_2, \ldots, x_n\|, \text{ for each } (x_1, x_2, \ldots, x_n) \in D(f).$$

If f is a bounded n–functional, we define a norm of f, denoting by $\|f\|$, with

$$\|f\| = \inf\{k\colon |f(x_1, x_2, \ldots, x_n)| \leq k\|x_1, x_2, \ldots, x_n\|,\ \forall(x_1, x_2, \ldots, x_n) \in D(f)\}.$$

If f is not a bounded n–functional, we put $\|f\| = +\infty$.

Lemma 1. Let $L, (\|\bullet, \ldots, \bullet\|)$ be a n–normed space and let f is a bounded n–functional with domain $D(f) \subset L^n$. If $x_i = \lambda x_j$, for $i, j \in \{1, \ldots, n\}$, $i \neq j$, and $(x_1, \ldots x_n) \in D(f)$, then

$$f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) = 0.$$

Proof. Since f is a bounded n–functional with domain $D(f)$, there is a real constant $k \geq 0$, such that for each $(x_1, x_2 \ldots x_n) \in D(f)$ it is true

$$
\begin{aligned}
&|f(x_1, x_2, \ldots, x_n)| \leq k\|x_1, x_2, \ldots, x_n\| = \\
&= k\|x_1, x_2, \ldots, x_{i-1}, \lambda x_j, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n\| = 0,
\end{aligned}
$$

which means $f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) = 0$.

Theorem 3. Let $(L, \|\bullet, \ldots, \bullet\|)$ be a n–normed space with dimension equal or smaller than $n+1$ and $\|x_1, \ldots, x_n\| > 0$. Then there exist a bounded n–linear functional F on L^n such that $\|F\| = 1$ and $F(x_1, \ldots, x_n) = \|x_1, \ldots, x_n\|$.

Proof. Since $\dim L \leq n+1$ there is a norm $\| \bullet \|$ on N_L such that

$$\|y_1, y_2, \ldots, y_n\| = \|N(y_1 \times y_2 \times \ldots \times y_n)\|, \text{ for every } y_1, y_2, \ldots, y_n \in L.$$

If we put $N' = N(x_1 \times x_2 \times \ldots \times x_n)$ and use the Han–Banach theorem on the space $(N_L, \| \bullet \|)$ we get a bounded linear functional f on N_L such that $\|f\| = 1$ and $f(N') = \|N'\|$. If we define

$$F(y_1, y_2, \ldots, y_n) = f\big(N(y_1 \times y_2 \times \ldots \times y_n)\big), \text{ for every } y_1, y_2, \ldots, y_n \in L$$

then F is a n–linear functional on L^n with the searching properties.

3. Strong n–convex n–normed space

Definition 3. We call a n–normed space $(L, \|\bullet, \ldots, \bullet\|)$ a strong n–convex if for every vectors $x_1, \ldots, x_{n+1} \in L$ which satisfy the conditions

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = \frac{1}{n+1} \|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = 1,$$

for $i = 1, 2, \ldots, n+1$ it is true $x_{n+1} = \sum_{i+1}^{n} x_i$.

Theorem 4. The following statements are equivalents:

i) $(L, \|\bullet, \ldots, \bullet\|)$ is strong n–convex.

ii) If $\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = 1$, for $i = 1, 2, \ldots, n+1$ and there is a non–zero bounded n–linear functional F on

$$\underbrace{P(x_1, \ldots, x_{n+1}) \times \ldots \times P(x_1, \ldots, x_{n+1})}_{n}$$

such that

$$F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) = \|F\|, \quad \text{for} \quad i = 1, 2, \ldots, n+1.$$

then $x_{n+1} = \sum_{i=1}^{n} x_i$.

iii) If

$$\|x_1 + x_{n+1},\, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = \sum_{i=1}^{n+1} \|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\|$$

and

$$\prod_{i=1}^{n+1} \|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| \neq 0\,,$$

then there are positive real numbers α_i, $i = 1, 2, \ldots, n$ such that

$$x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i\,.$$

Proof. i) $\Rightarrow$ ii). Let $(L, \|\bullet, \ldots, \bullet\|)$ be a strong n–convex,

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = 1, \quad \text{for} \quad i = 1, 2, \ldots, n+1$$

and there is a non–zero bounded n–linear functional F on

$$\underbrace{P(x_1, \ldots, x_{n+1}) \times \ldots \times P(x_1, \ldots, x_{n+1})}_{n}$$

such that

$$F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1} = \|F\|, \quad \text{for} \quad i = 1, 2, \ldots, n+1\,.$$

Since

$$\begin{aligned} n+1 &= \sum_{i=1}^{n+1} \|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| \geq \\ &\geq \|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| \geq \\ &\geq \frac{1}{\|F\|} F(x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}) = \\ &= \frac{1}{\|F\|} \sum_{i=1}^{n+1} F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}) = n+1 \end{aligned}$$

we have

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = \frac{1}{n+1} \|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = 1,$$

for $i = 1, 2, \ldots, n+1$. Because $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex we have $x_{n+1} = \sum_{i=1}^{n} x_i$.

ii) $\Rightarrow$ iii). Suppose that the condition ii) is true and that

$$\|x_1+x_{n+1}, x_2+x_{n+1}, \dots, x_n+x_{n+1}\| = \sum_{i=1}^{n+1} \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_{n+1}\|,$$

$$\prod_{i=1}^{n+1} \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_{n+1}\| \neq 0 .$$

According to theorem 3, there is a bounded n–linear functional F on $\underbrace{P(x_1, \dots, x_{n+1}) \times \dots \times P(x_1, \dots, x_{n+1})}_{n}$ such that $\|F\| = 1$ and

$$F(x_1+x_{n+1}, x_2+x_{n+1}, \dots x_n+x_{n+1}) = \|x_1+x_{n+1}, x_2+x_{n+1}, \dots, x_n+x_{n+1}\| .$$

Since $\|F\| = 1$, we have

$$\sum_{i=1}^{n+1} \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_{n+1}\| \geq$$

$$\geq \sum_{i=1}^{n+1} F(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_{n+1}\| =$$

$$= F(x_1 + x_{n+1}, x_2 + x_{n+1}, \dots, x_n + x_{n+1}) =$$

$$= \|x_1 + x_{n+1}, x_2 + x_{n+1}, \dots, x_n + x_{n+1}\| =$$

$$= \sum_{i=1}^{n+1} \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_{n+1}\|$$

Hence, $F(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) = \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}\|$, for $i = 1, 2, \dots, n+1$. Let

$$t = \left(\prod_{i=1}^{n+1} \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}\| \right)^{-1/n},$$

$$\beta_i = \|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}\| t, \quad \text{for} \quad i = 1, 2, \dots, n+1 .$$

Then

$$\|\beta_1 x_1, \dots, \beta_{i-1} x_{i-1}, \beta_{i+1} x_{i+1}, \dots, \beta_{n+1} x_{n+1}\| = 1, \text{ for } i = 1, 2, \dots, n+1$$

and

$$F(\beta_1 x_1, \dots, \beta_{i-1} x_{i-1}, \beta_{i+1} x_{i+1}, \dots, \beta_{n+1} x_{n+1}) = 1 = \|F\|, \text{ for } , i=1, 2, \dots, n+1 .$$

It follows from ii) that $\beta_{n+1}x_{n+1} = \sum_{n=1}^{n} \beta_i x_i$ and if we put $\alpha_i = \frac{\beta_i}{\beta_{n+1}}$, $i = 1, 2, \ldots, n$, we get $x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i$.

iii) $\Rightarrow$ i). Assume that the condition iii) is true and that

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n + x_{n+1}\| = \frac{1}{n+1} \|x_1+x_{n+1}, x_2+x_{n+1}, \ldots, x_n+x_{n+1}\| = 1,$$

for $i=1,2,\ldots,n+1$. Then, there exists positive real numbers α_i, $i=1,2,\ldots,n$ such that $x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i$. Since $\|x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = 1$, for $i = 1, 2, \ldots, n+1$ we get $\alpha_i = 1$, for $i = 1, 2, \ldots, n$ e.t. $x_{n+1} = \sum_{i=1}^{n} x_i$, which means that $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space.

In the next two theorems we will view the conection between the strong convex and n–strong convex n–normed spaces. If a, b are lineary independent vectors, then with $P(a, b)$ we will denote the space generated by the vectors a and b. Simillary, if $x_1, \ldots, x_{n-1}$ are lineary independent vectors, $P(x_1, \ldots, x_{n-1})$ means the subspace generated by this vectors. We call the n–normed vector space $(L, \|\bullet, \ldots, \bullet\|)$ strong convex if

$$\|a + b, x_1, \ldots, x_{n-1}\| = \|a, x_1, \ldots, x_{n-1}\| + \|b, x_1, \ldots, x_{n-1}\|,$$
$$\|a, x_1, \ldots, x_{n-1}\| = \|b, x_1, \ldots, x_{n-1}\| = 1$$

and

$$P(a, b) \cap P(x_1, \ldots, x_{n-1}) = \{0\}$$

implies $a = b$.

Theorem 5. If $(L, \|\bullet, \ldots, \bullet\|)$ is a strong convex space, then it is a strong n–convex space.

Proof. Suppose that the vectors $x_1, \ldots, x_{n+1} \in L$ satisfy the conditions

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = \frac{1}{n+1} \|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = 1,$$

for $i = 1, 2, \ldots, n+1$. Then,

$$\begin{aligned}
n+1 =&\|x_1+x_{n+1},\, x_2+x_{n+1},\, \ldots,\, x_n+x_{n+1}\| \leq \\
&\leq \|x_1,\, x_2+x_{n+1},\, \ldots,\, x_n+x_{n+1}\|+ \\
&\qquad + \|x_{n+1},\, x_2+x_{n+1},\, \ldots,\, x_n+x_{n+1}\| = \\
&=1+\|x_1,\, x_2+x_{n+1},\, \ldots,\, x_n+x_{n+1}\| \leq \\
&\leq 1+\|x_1,\, x_2,\, x_3+x_{n+1},\, \ldots,\, x_n+x_{n+1}\|+ \\
&\qquad + \|x_1,\, x_{n+1},\, x_3+x_{n+1},\, \ldots,\, x_n+x_{n+1}\| = \\
&=2+\|x_1,\, x_2,\, x_3+x_{n+1},\, \ldots,\, x_n+x_{n+1}\| \leq \cdots \leq \\
&\leq n-1+\|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_n+x_{n+1}\|
\end{aligned}$$

which means that $2 \leq \|x_1, x_2, x_3, \ldots, x_{n-1}, x_n + x_{n+1}\|$. For the other side, we have:

$$\begin{aligned}
&\|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_n+x_{n+1}\| \leq \|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_n\|+ \\
&\quad +\|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_{n+1}\| = 2\,.
\end{aligned}$$

Hence,

$$\|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_n+x_{n+1}\| = 2$$

and

$$\|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_n\| = \|x_1,\, x_2,\, x_3,\, \ldots,\, x_{n-1},\, x_{n+1}\| = 1\,.$$

Since $(L, \|\bullet, \ldots, \bullet\|)$ is a strong convex and $x_n \neq x_{n+1}$ we have

$$P(x_1,\, \ldots,\, x_{n-1}) \cap P(x_n,\, x_{n+1}) \neq \{0\}\,,$$

which means that there exists real numbers α_i, $i = 1, 2, \ldots, n+1$ such that

$$\sum_{i=1}^{n-1} \alpha_i x_i = \alpha_n x_n + \alpha_{n+1} x_{n+1}\,. \tag{1}$$

If $\alpha_k = 0$, for some $k \in \{1, 2, \ldots, n+1\}$, then

$$\|x_1,\, \ldots,\, x_{k-1},\, x_{k+1},\, \ldots,\, x_n,\, x_{x+1}\| = 0,$$

which im impossible. Hence, $\alpha_k \neq 0$, for every $k \in \{1, 2, \ldots, n+1\}$. Now, from (1) we have

$$x_{n+1} = \sum_{i=1}^{n} \beta_i x_i, \text{ where } \beta_n = -\frac{\alpha_n}{\alpha_{n+1}} \text{ and } \beta_i = -\frac{\alpha_i}{\alpha_{n+1}}\,, \text{ for } \; i=1,\, 2,\, \ldots,\, n-1\,.$$

If we substitute x_{n+1} in

$$\|x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,x_{n+1}\|=\frac{1}{n+1}\|x_1+x_{n+1},x_2+x_{n+1},\dots,x_n+x_{n+1}\|=1,$$

for $i=1,2,\dots,n$ then by the properties of the n–norm, we get

$$1=|\beta_i|\cdot\|x_1-x_n,\dots,x_{n-1}-x_n,\,x_n\|=\frac{\left|1+\sum_{i=1}^{n}\beta_i\right|}{n+1}\|x_1-x_n,\dots,x_{n-1}-x_n,\,x_n\|$$

for $i=1,2,\dots,n$ and since

$$1=\|x_1,\,\dots,\,x_{n-1},\,x_n\|=\|x_1-x_n,\,\dots,\,x_{n-1}-x_n,\,x_n\|$$

we have

$$|\beta_i|=1,\quad \text{for}\quad i=1,2,\dots,n\quad \text{and}\quad \left|1+\sum_{i=1}^{n}\beta_i\right|=n+1\,. \tag{2}$$

From (2) we have $\beta_i=1$, for $i=1,2,\dots,n$, e.t. $x_{n+1}=\sum_{i=1}^{n}x_i$, which mens that $(L,\|\bullet,\dots,\bullet\|)$ is a strong n–convex.

Theorem 6. Let $(L,\|\bullet,\dots,\bullet\|)$ be a n-normed space and $(L_1,\|\bullet,\dots,\bullet\|)$ is a strong n–convex n–normed space. If $f\colon L\to L_1$ is a linear mapping such that

$$\|f(x_1),\,\dots,\,f(x_n)\|'=\|x_1,\,\dots,\,x_n\|\quad\text{for every}\quad x_1,\dots,x_n\in L\,,$$

then f is injection and $(L,\|\bullet,\dots,\bullet\|)$ is a strong n-convex. If $(L_1,\|\bullet,\dots,\bullet\|)$ is also strong convex, then $(L,\|\bullet,\dots,\bullet\|)$ is a strong convex space.

Proof. For every $x_1\neq 0$ there exist $x_2,\dots,x_n\in L$ such that the set $\{x_1,x_2,\dots,x_n\}$ is lineary independent. Hence, $\|f(x_1),\dots,f(x_n)\|'=\|x_1,\dots,x_n\|\neq 0$ e.t. $f(x_i)\neq 0$. Since f is a linear mapping, this means that f is a injection.

Suppose that the $x_1,\dots,x_{n+1}\in L$ satisfies the conditions:

$$\|x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,x_{i+1}\|=\frac{1}{n+1}\|x_1+x_{n+1},x_2+x_{n+1},\dots,x_n+x_{n+1}\|=1,$$

for i=1, 2, ..., n+1. Then, for i=1, ..., n+1, the vectors $f(x_1), \ldots, f(x_{n+1}) \in L_1$ satisfies

$$\|f(x_1), \ldots, f(x_{i-1}), f(x_{i+1}), \ldots, f(x_n), f(x_{n+1})\|' =$$
$$= \frac{1}{n+1} \|f(x_1) + f(x_{n+1}), \ldots, f(x_n) + f(x_{n+1})\|' = 1 .$$

Since L_1 is a strong n–convex space and f is a linear mapping, we have

$$f(x_{n+1}) = \sum_{i=1}^{n} f(x_i) = f\left(\sum_{i=1}^{n} x_i\right) .$$

But, f is a injection and so $x_{n+1} = \sum_{i=1}^{n} x_i$, which means that $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space.

Suppose that

$$P(a,b) \cap P(x_1, x_2, \ldots, x_{n-1} = \{0\} ,$$

$$\|a, x_1, x_2, \ldots, x_{n-1}\| = \|b, x_1, x_2, \ldots, x_{n-1}\| = 1 \qquad \text{and}$$
$$\|a+b, x_1, x_2, \ldots, x_{n-1}\| = \|a, x_1, x_2, \ldots, x_{n-1}\| + \|b, x_1, x_2, \ldots, x_{n-1}\| .$$

We have

$$\|f(a) + f(b), f(x_1), \ldots, f(x_{n-1})\|' =$$
$$= \|f(a), f(x_1), \ldots, f(x_{n-1})\|' + \|f(b), f(x_1), \ldots, f(x_{n-1})\|'$$

and

$$\|f(a), f(x_1), \ldots, f(x_{n-1})\|' = \|f(b), f(x_1), \ldots, f(x_{n-1})\|' = 1.$$

If $y \in P\big(f(a), f(b)\big) \cap P\big(f(x_1), \ldots, f(x_{n-1})\big)$, then there exists $\lambda, \mu, \alpha_i, i = 1, \ldots, n-1$ such that $y = \lambda f(a) + \mu f(b) = \sum_{i=1}^{n-1} \alpha_i f(x_i)$. Hence, $f(\lambda a + \mu b) = f\left(\sum_{i=1}^{n-1} \alpha_i x_i\right)$ and since f is a injection it follows that $\lambda a + \mu b = \sum_{i=1}^{n-1} \alpha_i x_i$. Since, $P(a, b) \cap P(x_1, x_2, \ldots x_{n-1}) = \{0\}$, we have

$$\lambda a + \mu b = \sum_{i=1}^{n-1} \alpha_i x_i = 0 , \qquad \text{e.t.} \qquad y = f(0) = 0 .$$

This implies that

$$P\big(f(a),\, f(b)\big) \cap P\big(f(x_1),\, \ldots,\, f(x_{n-1})\big) = \{0\}$$

and because L_1 is a strong convex space, we have $f(a) = f(b)$. But f is a injection and so $a = b$, which means that $(L, \|\bullet, \ldots, \bullet\|)$ is a strong convex space.

In the end od this part we will give one more condition for a strong convexity. First, we will view one property of the normed spaces.

Definition 4. The normed space $(X, \|\bullet\|)$ has the property C_n if

$$\|x_1\| = \cdots = \|x_{n+1}\| = \frac{1}{n+1}\,\|x_1 + \cdots + x_{n+1}\| = 1$$

implies that the vectors $x_1, \ldots, x_{n+1}$ are colinear.

Lemma 2. If the normed space $(X, \|\bullet\|)$ is a strong convex, then for every $n \in N$ it has property C_n.

Proof Let $n \in N$. We have

$$\left\|\frac{1}{n}(x_2 + \cdots + x_{n+1})\right\| \le \frac{1}{n}\sum_{i=2}^{n+1} \|x_i\| = 1\,.$$

On the other side

$$1 = \frac{1}{n+1}\,\|x_1 + \cdots + x_{n+1}\| \le \frac{1}{n+1}\left(\|x_1\| + n\left\|\frac{1}{n}(x_2 + \cdots + x_{n+1})\right\|\right)$$

which means

$$n+1 \le \|x_1\| + n\left\|\frac{1}{n}(x_2 + \cdots + x_{n+1})\right\| \quad \text{e.t.} \quad 1 \le \left\|\frac{1}{n}(x_2 + \cdots + x_{n+1})\right\|.$$

Hence

$$\left\|\frac{1}{n}(x_2 + \cdots + x_{n+1})\right\| = 1\,.$$

Since $(X, \|\bullet\|)$ ia a strong convex,

$$\begin{aligned} &\|x_1 + x_2 + \cdots + x_{n+1}\| = n+1 = \|x_2 + \cdots + x_{n+1}\| + \|x_1\|\,, \\ &\qquad x_1 \ne 0, \quad x_2 + \cdots + x_{n+1} \ne 0 \end{aligned}$$

implies $x_2 + \cdots + x_{n+1} = \alpha x_1$, for some $\alpha > 0$. But,

$$n+1 = \|x_1 + x_2 + \cdots + x_{n+1}\| = \|x_1 + \alpha x_1\| = (1+\alpha)\|x_1\| = 1 + \alpha, \quad \text{e.t. } \alpha = n,$$

and so $x_1 = \frac{1}{n}(x_2 + \cdots + x_{n+1})$. In the same way can be proved that each of the vectors $x_1, x_2, \ldots, x_{n+1}$ is a arithmetical mean of the rest n vectors, which implies

$$x_1 = x_2 = \cdots = x_{n+1},$$

and this mean, that for every $n \in N$ the space $(X, \|\bullet\|)$ has the property C_n.

Note. It easy to prove that the normed space $(X, \|\bullet\|)$ with the property C_1 is a strong convex space. But, for $n = 2$ there is an example in [4] of a normed space with the property C_2, but the space is not a strong convex.

Theorem 7. Let $(L, \|\bullet, \ldots, \bullet\|)$ be a n–normed space and $(X, \|\bullet\|)$ is a normed space with property C_n. If there is a mapping $T \in T_{L,X}$ such that

$$\|x_1, \ldots, x_n\| = \|T(x_1, \ldots, x_n)\|, \quad \text{for every} \quad x_1, \ldots, x_n \in L,$$

then $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space.

Proof. Suppose that the vectors $x_1, \ldots, x_{n+1} \in L$ satisfy the conditions

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = \frac{1}{n+1}\|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = 1,$$

if $i = 1, 2, \ldots, n+1$. Then, for $i = 1, 2, \ldots, n+1$ it is true

$$1 = \|T(x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n)\| =$$
$$= \|T(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1})\| =$$
$$= \frac{1}{n+1}\|T(x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1})\| =$$
$$= \frac{1}{n+1}\|\sum_{i=1}^{n+1} T(x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n)\|.$$

Since the normed space $(X, \|\bullet|)$ has the property C_n, the vectors

$$T(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}), \quad i = 1, 2, \ldots, n+1$$

are colinear.

If $T(x_1, \ldots, x_n) = T(x_1, \ldots, x_{n-1}, x_{n+1})$, then

$$\|x_1, \ldots, x_{n-1}, x_n - x_{n+1}\| = \|T(x_1, \ldots, x_{n-1}, x_n - x_{n+1}\| =$$
$$= \|T(x_1, \ldots, x_{n-1}, x_n) - T(x_1, \ldots, x_{n-1}, x_{n+1})\| = 0$$

and

$$\|x_1, \ldots, x_{n-1}, x_n\| = 1$$

implies

$$x_{n+1} = x_n - \sum_{i=1}^{n-1} \alpha_i x_i \,.$$

For every $i \in \{1, \ldots, n-1\}$ it is true

$$\begin{aligned}
1 &= \|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = \\
&= \left\| x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_n - \sum_{j=1}^{n-1} \alpha_j x_j \right\| = \\
&= |\alpha_i| \, \|x_1, \ldots, x_n\| = |\alpha_i|
\end{aligned}$$

and

$$\begin{aligned}
1 &= \frac{1}{n+1} \|x_1 + x_{n+1}, x_2 + x_{n+1} \ldots, x_n + x_{n+1}\| = \\
&= \frac{1}{n+1} \|x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n, x_n + x_{n+1}\| = \\
&= \frac{1}{n+1} \left\| x_1 - x_n, \ldots, x_{n-1} - x_n, 2x_n - \sum_{i=1}^{n-1} \alpha_i x_i \right\| = \\
&= \frac{1}{n+1} \left\| x_1 - x_n, \ldots, x_{n-1} - x_n, x_n \left(2 - \sum_{i=1}^{n-1} \alpha_i \right) \right\| = \\
&= \frac{1}{n+1} \left| 2 - \sum_{i=1}^{n-1} \alpha_i \right| \, \|x_1 - x_n, \ldots, x_{n-1} - x_n, x_n\| = \\
&= \frac{1}{n+1} \left| 2 - \sum_{i=1}^{n-1} \alpha_i \right| \, \|x_1, \ldots, x_n\| = \frac{1}{n+1} \left| 2 - \sum_{i=1}^{n-1} \alpha_i \right| .
\end{aligned}$$

From $|\alpha_i| = 1$, $i \in \{1, \ldots, n-1\}$ and $\left| 2 - \sum_{i=1}^{n-1} \alpha_i \right| = n+1$, it follows that $\alpha_i = -1$, $i = 1, \ldots, n-1$ which means that $x_{n+1} = \sum_{i=1}^{n} x_i$, e.t. $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex.

Suppose that there is no two points which are identical, e.t. that

$$T(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i T(x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n).$$

So

$$T(x_1 - \alpha_1 x_{n+1}, \ldots, x_n - \alpha_n x_{n+1}) = 0,$$

and hence

$$\|x_1 - \alpha_1 x_{n+1}, \ldots, x_n - \alpha_n x_{n+1}) = \|T(x_1 - \alpha_1 x_{n+1}, \ldots, x_n - \alpha_n x_{n+1})\| = 0$$

which implies that there exists real numbers β_i, $i = 1, 2, \ldots, n$ not all equal to zero and such that

$$\sum_{i=1}^{n} \beta_i x_i = x_{n+1} \sum_{i=1}^{n} \alpha_i \beta_i.$$

It is clear that, $\sum_{i=1}^{n} \alpha_i \beta_i \neq 0$ and if we put $\gamma_i = \dfrac{\beta_i}{\sum_{i=1}^{n} \alpha_i \beta_i}$ we get

$x_{n+1} = \sum_{i=1}^{n} \gamma_i x_i$. As in the previous case can be proved that $\gamma_i = 1$, $i = 1, 2, \ldots, n$ e.t. $x_{n+1} = \sum_{i=1}^{n} x_i$, which means that $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex.

4. Algebraic and n–norm middle points

Let $x_1, x_2 \ldots, x_n \in L$. Denote with $P(x_1, x_2 \ldots, x_n)$ the subspace generated by the vectors $x_1, x_2 \ldots, x_n$.

Definition 5. Suppose that $(L, \|\bullet, \ldots, \bullet\|)$ is a n–normed space. The point $a \in L$ we call algebraic middle point of the points $x_1, x_2 \ldots, x_k \in L$ if $a = \dfrac{1}{k} \sum_{i=1}^{k} x_i$. The point $a \in L$ we call n–normed middle point of the $n + 1$ lineary independent vectors $x_1, x_2 \ldots, x_n, x_{n+1} \in L$ if

$$\|x_1-a,\ldots,x_{i-1}-a,x_{i+1}-a,\ldots,x_{n+1}-a\|=\frac{1}{n+1}\|x_1-x_{n+1},x_2-x_{n+1}\cdots,x_n-x_{n+1}\|,$$

for $i = 1, 2, \ldots, n+1$.

Theorem 8. If $a \in L$ is a algebraic middle point of $n+1$ lineary independent vectors $x_1, x_2 \ldots, x_n, x_{n+1} \in L$, then it is a n–norm middle point of this vectors.

Proof. If $a = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i$, then for every $i = 1, 2, \ldots, n, n+1$ it is true that

$$\|x_1-a,\ldots,x_{i-1}-a,x_{i+1}-a,\ldots,x_{n-1}-a,x_n-a,x_{n+1}-a\|=$$

$$=\left\|x_1-\frac{1}{n+1}\sum_{j=1}^{n+1}x_j,\ldots,x_{i-1}-\frac{1}{n+1}\sum_{j=1}^{n+1}x_j,x_{i+1}-\frac{1}{n+1}\sum_{j=1}^{n+1}x_j,\ldots,x_n-\frac{1}{n+1}\sum_{j=1}^{n+1}x_j,x_{n+1}-\frac{1}{n+1}\sum_{j=1}^{n+1}x_j\right\|=$$

$$=\frac{1}{(n+1)^n}\left\|(n+1)x_1-\sum_{j=1}^{n+1}x_j,\ldots,(n+1)x_{i-1}-\sum_{j=1}^{n+1}x_j,(n+1)x_{i+1}-\sum_{j=1}^{n+1}x_j,\ldots,(n+1)x_n-\sum_{j=1}^{n}x_j,(n+1)x_{n+1}-\sum_{j=1}^{n+1}x_j\right\|=$$

$$=\frac{1}{(n+1)^n}\left\|(n+1)(x_1-x_{n+1}),\ldots,(n+1)(x_{i-1}-x_{n+1}),(n+1)(x_{i+1}-x_{n+1}),\ldots,(n+1)(x_n-x_{n+1}),nx_{n+1}-\sum_{j=1}^{n}x_j\right\|=$$

$$=\frac{1}{n+1}\left\|x_1-x_{n+1},\ldots,x_{i-1}-x_{n+1},x_{i+1}-x_{n+1},\ldots,x_n-x_{n+1},nx_{n+1}-\sum_{j=1}^{n}x_j\right\|=$$

$$=\frac{1}{n+1}\|x_1-x_{n+1},\ldots,x_{i-1}-x_{n+1},x_{i+1}-x_{n+1},\ldots,x_n-x_{n+1},x_{n+1}-x_i\|=$$

$$=\frac{1}{n+1}\|x_1-x_{n+1},\ldots,x_{i-1}-x_{n+1},x_i-x_{n+1},x_{i+1}-x_{n+1},\ldots,x_n-x_{n+1}\|,$$

which means that a is a n–norm middle point of the vectors $x_1, x_2, \ldots, x_n, x_{n+1} \in L$.

Theorem 9. If $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space, then every n–norm middle point of $n+1$ lineary independent vectors is an algebraic middle point of this vectors.

Proof. Suppose that $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–conex space and that a is a n–norm middle point lineary independent vectors $x_1, \ldots, x_{n+1}$.

If we put

$$t^n = \|x_1 - a, \ldots, x_{i-1} - a, x_{i+1} - a, \ldots, x_{n+1} - a\| = $$
$$= \frac{1}{n+1} \|x_1 - x_{n+1}, x_2 - x_{n+1}, \ldots, x_n - x_{n+1}\|,$$

then since $x_1, \ldots, x_{n+1}$ are lineary independent vectors, we have $t \neq 0$, and hence

$$1 = \left\| \frac{x_1 - a}{t}, \ldots, \frac{x_{i-1} - a}{t}, \frac{x_{i+1} - a}{t}, \ldots, \frac{a - x_{n+1}}{t} \right\| =$$
$$= \frac{1}{n+1} \left\| \frac{x_1 - x_{n+1}}{t}, \frac{x_2 - x_{n+1}}{t}, \ldots, \frac{x_n - x_{n+1}}{t} \right\| =$$
$$= \left\| \frac{x_1 - a}{t} + \frac{a - x_{n+1}}{t}, \frac{x_2 - a}{t} + \frac{a - x_{n+1}}{t}, \ldots, \frac{x_n - a}{t} + \frac{a - x_{n+1}}{t} \right\|,$$

for $i = 1, 2, \ldots, n+1$.

But since $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space, we have

$$\frac{a - x_{n+1}}{t} = \sum_{i=1}^{n} \frac{x_i - a}{t}, \qquad \text{e.t.} \qquad a = \frac{1}{n+1} \sum_{i=1}^{n+1} x_i .$$

Theorem 10. The n–normed space $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex if and only if for every $n+1$ lineary inpedent vectors the algebraic and n– norm middle point are identical.

Proof. If $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space, then the theorems 3 and 4 implies that the algebraic and n–norm middle points of arbitrary $n+1$ lineary independent vectors are identical.

Suppose that in $(L, \|\bullet, \ldots, \bullet\|)$ algebraic and n–norm middle points of arbitrary $n+1$ linear independent vectors are identical. From

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\| = \frac{1}{n+1} \|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = 1,$$

if $i = 1, 2, \ldots, n+1$ it follows that

$$\|x_1 - 0, \ldots, x_{i-1} - 0, x_{n+1} - 0, \ldots, x_{n+1} - 0\| = \frac{1}{n+1} \|x_1 + x_{n+1}, x_2 + x_{n+1}, \ldots, x_n + x_{n+1}\| = 1,$$

if $i = 1, 2, \ldots, n+1$ which means that 0 is a n-norm middle point of the vectors $x_1, \ldots, x_n, -x_{n+1}$. Hence, 0 is a algebraic middle point of the vectors $x_1, \ldots, x_n, -x_{n+1}$, which means

$$0 = \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i - x_{n+1}\right), \qquad \text{e.t.} \qquad x_{n+1} = \sum_{i=1}^{n} x_i, ,$$

and so $(L, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex space.

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СТРОГО n–КОНВЕКСНИ n–НОРМИРАНИ ПРОСТОРИ

Ристо Малчески

Резиме

Концептот за n скаларен производ на векторски простор со димензија поголема од $n-1$ и n–норма, воведен од A. Misiak ([6]) е повеќедимензионална аналогија на концептот за скаларен производ и норма. Во [6] и [5] се докажани основните својства на n–предхилбертов и n–нормиран простор. Во оваа работа е дадена генерализација на поимот строго 2–конвексен 2–нормиран простор, разгледуван во [2] и [3] и се докажани низа својства на строго n–конвексните n–нормирани простори.

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