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REDUCTIONS FOR PRESENTATIONS OF (n,m)–SEMIGROUPS INDUCED BY REDUCTIONS FOR PRESENTATIONS OF BINARY SEMIGROUPS

DONČO DIMOVSKI AND IRENA STOJMENOVSKA

Abstract. The question of finding a satisfactory combinatorial description of an (n,m)–semigroup given with its (n,m)–presentation $\langle B;\Delta\rangle$ can be answered by managing to construct a good reduction for the given $\langle B;\Delta\rangle$ (if possible), which is usually quite complicated to achieve. Here, we construct good reductions for a class of (n,m)–presentations of (n,m)–semigroups that are induced by presentations of binary semigroups satisfying certain conditions. Namely, given a semigroup presentation $\langle B;\Lambda\rangle$ with a good reduction φ that satisfy a pair of conditions, we define an associated (n,m)–semigroup presentation $\langle B;\Delta\rangle$ and derive a good reduction ψ for $\langle B;\Delta\rangle$. As a consequence, good description of the corresponding (n,m)–semigroup is obtained.

1. Introduction and preliminaries

Bellow we give some definitions, notations and facts on combinatorial semigroup theory and combinatorial (n,m)–semigroup theory. (For more details see [2], [4]).

Let B be a nonempty set and let $\boldsymbol{B}^+$ be the free semigroup with basis B. $\boldsymbol{B}^+ = (B^+;\cdot)$ where B^+ is the set of all finite (nonempty) sequences of the elements of B and '$\cdot$' is the concatenation of sequences. The element $(a_1,\ldots,a_r)\in B^r\subseteq B^+$ will be denoted simply by a_1^r, or by $\overset{r}{a}$ in the case when $a_1=\ldots=a_r=a$. Also, a_i^j will denote the sequence $a_ia_{i+1},\ldots,a_j$ when $i\leq j$ or the empty sequence when $i>j$. Sometimes $\underline{x}$ will be a short notation for a sequence of elements of a set B. As usual, d will be used to denote the dimension of a sequence $a_1^r\in B^r$ (i.e. $d(a_1^r)=r$), and $\mathbb{N}$ will denote the set of all positive integers. By $\mathbb{N}_r$ and $\mathbb{N}_r^0$ we denote the sets {1,2, ..., r} and {0,1, ..., r} respectively, where $r\in\mathbb{N}$.

Let $\Lambda\subseteq B^+\times B^+$. The pair $\langle B;\Lambda\rangle$ is a presentation of the semigroup $\boldsymbol{B}^+/\Lambda^=$ where $\Lambda^=$ is the smallest congruence on $\boldsymbol{B}^+$ containing Λ. We use the notation $\langle B;\Lambda\rangle=\boldsymbol{B}^+/\Lambda^=$.

A reduction for $\langle B;\Lambda\rangle$ is a mapping $\varphi: B^+\to B^+$ satisfying the conditions:
(i) $\varphi(xuy)=\varphi(x\varphi(u)y)$, (ii) $(u,v)\in\Lambda\Rightarrow\varphi(u)=\varphi(v)$, (iii) $\varphi(u)\Lambda^= u$,
for all $u,v\in B^+$ and $x,y\in B^*$, where $B^*=B^+\cup\{1\}$ and 1 is a notation for the empty sequence.

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Each reduction φ for $\langle B;\Lambda\rangle$ is a homomorphism from $\boldsymbol{B}^+$ to $(\varphi(B^+);\circ)$, where the operation '$\circ$' on $\varphi(B^+)$ is defined by $u\circ v=\varphi(uv)$, $u,v\in\varphi(B^+)$. Moreover, $\ker\varphi=\Lambda^=$ and thus $\boldsymbol{B}^+/\Lambda^=\cong(\varphi(B^+);\circ)$ i.e. $\langle B;\Lambda\rangle=(\varphi(B^+);\circ)$.

A reduction φ for $\langle B;\Lambda\rangle$ is good (effective) reduction for $\langle B;\Lambda\rangle$ if there exists an invariant $\rho: B^+\to\mathbb{N}$ such that $\varphi(x)\neq x$ implies $\rho(\varphi(x))<\rho(x)$ for all $x\in B^+$. In this case, for a given $u\in B^+$, the reduced represent $\varphi(u)$ can be determined in a finite number of steps. (As a consequence, the existence of an algorithm for the decidability i.e. solvability of the word problem is provided).

Let $Q\neq\emptyset$, $n,m\in\mathbb{N}$ and let $n-m=k\geq 1$. We will also assume that $m\geq 2$. A mapping $f: Q^n\to Q^m$ is an (n,m)–operation and the pair $\boldsymbol{Q}=(Q;f)$ is called an (n,m)–groupoid. A mapping $f:\bigcup\limits_{s\geq 1}Q^{m+sk}\to Q^m$ is called a poly-(n,m)–operation and the pair $\boldsymbol{Q}=(Q;f)$ is said to be a poly-(n,m)–groupoid.

An (n,m)–groupoid $\boldsymbol{Q}=(Q;f)$ is an (n,m)–semigroup if

$$f(f(x_1^n)x_{n+1}^{n+k})=f(x_1^i f(x_{i+1}^{i+n})x_{i+n+1}^{n+k})\ \text{ for all } x_v\in Q,\ i\in\mathbb{N}_k.$$

A poly-(n,m)–groupoid $\boldsymbol{Q}=(Q;f)$ is a poly-(n,m)–semigroup if

$$f(x_1^j f(y_1^{m+rk})x_{j+1}^{sk})=f(x_1^j y_1^{m+rk}x_{j+1}^{sk})\ \text{ for all } x_\lambda,y_\mu\in Q,\ s,r\geq 1,\ j\in\mathbb{N}_{sk}^0.$$

Remark. It is not necessary to make distinction between the notions of (n,m)–semigroup and poly-(n,m)–semigroup due to the fact there is no essential difference between them, a consequence from the general associative law (GAL) which holds in all (n,m)–semigroups. (See [2], §5.)[3]

The notions of (n,m)–operations (poly-(n,m)–operations) are easily thought of as algebras with m n-ary (poly n-ary) operations

$$f_1,\ldots,f_m:\bigcup_{s\geq 1}Q^{m+sk}\to Q,\ \text{ where } f_i(x_1^{m+sk})\overset{def}{=}z_i\Leftrightarrow f(x_1^{m+sk})=z_1^m,\ i\in\mathbb{N}_m,$$

and $s=1$ for the (n,m)–case (i.e. $s\geq 1$ for the poly-(n,m)–case).

This allow us to translate all the notions which make sense for universal algebras to [poly-](n,m)–goupiods, without giving their explicit definitions.

Let $\boldsymbol{F(B)}=(F(B);f)$ be a free poly-(n,m)–groupoid with a basis B. We recall its construction. (See [2], §6).

$$B_{-1}=\emptyset,\quad B_0=B,\quad B_{p+1}=B_p\cup\big(\mathbb{N}_m\times\bigcup_{s\geq 1}B_p^{m+sk}\big),\quad F(B)=\bigcup_{p\geq 0}B_p.$$

The poly-(n,m)–operation f on $F(B)$ is defined by

$$f(u_1^{m+sk})=v_1^m\Leftrightarrow(\forall i\in\mathbb{N}_m)\,v_i=(i,u_1^{m+sk}).$$

Hierarchy of the elements of $F(B)$ is a mapping $\chi: F(B)\to\mathbb{N}_0$ defined by
$\chi(u)=\min\{p\,|\,u\in B_p\}$. Clearly, $\chi(u)=p\Leftrightarrow u\in B_p\backslash B_{p-1}$.

Length on $F(B)$ is a mapping $|\ |: F(B)\to\mathbb{N}$ defined by induction on χ:
$|u|=1$ for $u\in B_0$, $|(i,u_1^{m+sk})|=|u_1|+\ldots+|u_{m+sk}|$ for $(i,u_1^{m+sk})\in B_{p+1}\backslash B_p$.

[3] However, there is a justified reason for introducing poly-(n,m)–semigroups in the combinatorial (n,m)–semigroup theory (see [2], §6).

Definition 1.1 ([4]). *Let $\Delta \subseteq F(B) \times F(B)$. Δ is said to be a set of (n,m)–defining relations on B and the pair $\langle B;\Delta\rangle$ is a presentation of an (n,m)–semigroup.*

Proposition 1.1 ([4]). *$\langle B;\Delta\rangle$ presents the factor (n,m)–semigroup $\boldsymbol{F(B)}/\overline{\Delta}$ where $\overline{\Delta}$ is the least congruence on $\boldsymbol{F(B)}$ such that $\Delta \subseteq \overline{\Delta}$ and $\boldsymbol{F(B)}/\overline{\Delta}$ is an (n,m)–semigroup. We use the notation $\langle B;\Delta\rangle = \boldsymbol{F(B)}/\overline{\Delta}$.*

Definition 1.2 ([4]). *Reduction for $\langle B;\Delta\rangle$ is a mapping $\psi : F(B) \to F(B)$ with the following properties:*

(i) $(u,v) \in \Delta \Rightarrow \psi(u) = \psi(v)$

(ii) $\psi(i, x'(1,y)(2,y)\dots(m,y)x'') = \psi(i, x'yx'')$

(iii) $\psi(i, x'wx'') = \psi(i, x'\psi(w)x'')$

(iv) $u\overline{\Delta}\psi(u)$

(v) $\psi(\psi(u)) = \psi(u)$,

for all $u, v, w, (i, x'wx''), (i, x'(1,y)(2,y)\dots(m,y)x'') \in F(B)$ and $x', x'' \in F(B)^*$.

Theorem 1.1 ([4]). *The reduction $\psi : F(B) \to F(B)$ for $\langle B;\Delta\rangle$ is a homomorphism from $\boldsymbol{F(B)}$ to $(\psi(F(B)); g)$ where*

$$\psi(F(B)) = \{u \in F(B) \mid \psi(u) = u\} \text{ and}$$

$$g(u_1^{m+sk}) = v_1^m \Leftrightarrow v_i = \psi(i, u_1^{m+sk}),\ i \in \mathbb{N}_m.$$

Moreover, $\ker\psi = \overline{\Delta}$ *and thus $\boldsymbol{F(B)}/\overline{\Delta} \cong (\psi(F(B)); g)$ i.e. $\langle B;\Delta\rangle = (\psi(F(B)); g)$.*

If ψ is a reduction for $\langle B;\Delta\rangle$ such that $\psi(u)$ can be determined in a finite number of steps for a given $u \in F(B)$, then ψ is said to be a good (effective) reduction for $\langle B;\Delta\rangle$. (It provides the existence of an algorithm for calculating the reduced represent $\psi(u)$, $u \in F(B)$).

In the case when $\Delta = \emptyset$, the pair $\langle B;\emptyset\rangle$ presents the free (n,m)–semigroup with a basis B and $\langle B;\emptyset\rangle = \boldsymbol{F(B)}/\approx$, where $\approx$ is the least congruence on $\boldsymbol{F(B)}$ such that $\boldsymbol{F(B)}/\approx$ is an (n,m)–semigroup. We recall its combinatorial description from [3]. Let $\psi_0 : F(B) \to F(B)$ be a mapping defined as follows:

$$\psi_0(b) = b, b \in B;$$

Assume that $u = (i, u_1^{m+sk}) \in F(B)$ and that $\psi_0(v) \in F(B)$ is well defined for all $v \in F(B)$ such that $|v| < |u|$. Moreover, assume that $\psi_0(v) \neq v$ implies $|\psi_0(v)| < |v|$. Then, $v_\lambda = \psi_0(u_\lambda)$ is well defined for all $\lambda \in \mathbb{N}_{m+sk}$ and thus $v = (i, v_1^{m+sk}) \in F(B)$. If there exists a $\lambda' \in \mathbb{N}_{m+sk}$ such that $v_{\lambda'} \neq u_{\lambda'}$ then $|v| < |u|$ and consequently define

$$\psi_0(u) = \psi_0(v);$$

If $v_\lambda = u_\lambda$ for all $\lambda \in \mathbb{N}_{m+sk}$ and if $u = (i, u_1^j(1, w_1^{m+rk})\dots(m, w_1^{m+rk})u_{j+m+1}^{m+sk})$ where $w_1^{m+rk} \in F(B)^{m+rk}, (r \geq 1)$ and j is the smallest such index, define

$$\psi_0(u) = \psi_0(i, u_1^j w_1^{m+rk} u_{j+m+1}^{m+sk}).$$

If u doesn't satisfy any of the conditions above, $\psi_0(u) \stackrel{def}{=} u$.
The mapping ψ_0 is well defined and it reduces the length, i.e. $\psi_0(u) \neq u$ implies $|\psi_0(u)| < |u|$, $u \in F(B)$.

Proposition 1.2 ([3])**.** *The mapping ψ_0 is a good reduction for $\langle B; \emptyset \rangle$.*

Remark 1.1: Note that the reduction ψ_0 does not change the first coordinate nor decreases the dimension of the second coordinate when mapping elements from $F(B) \backslash B$. Also, ψ_0 does not increase the hierarchy i.e. $\chi(\psi_0(u)) \leq \chi(u)$, $u \in F(B)$. (The proofs are by induction on $|\ |$ and applying the definition of ψ_0.)

It is natural to look for a suitable combinatorial description of an (n, m)–semigroup given with its (n, m)–presentation $\langle B; \Delta \rangle$. Such description can be obtained if we manage to construct a good reduction for $\langle B; \Delta \rangle$, a task which is not easy nor always possible to fulfill. Some examples, constructions and results on the issue are given in [4], [5], [6]. Bellow we define a class of presentations of (n, m)–semigroups $\langle B; \Delta \rangle$ such that good reductions for $\langle B; \Delta \rangle$ can be constructed. These $\langle B; \Delta \rangle$ are induced by presentations of binary semigroups $\langle B; \Lambda \rangle$ with good reductions φ satisfying a pair of conditions. Given such (n, m)–semigroup presentation $\langle B; \Delta \rangle$, and using the good reduction φ for $\langle B; \Lambda \rangle$, as well as the good reduction ψ_0 for $\langle B; \emptyset \rangle$, we will construct a good reduction ψ for $\langle B; \Delta \rangle$.

2. Main part

Let $\langle B; \Lambda \rangle$ be a semigroup presentation satisfying the conditions:

(I') $d(\underline{x}), d(\underline{z}) > m$ for all $(\underline{x}, \underline{z}) \in \Lambda$

(II') There exists a good reduction $\varphi : B^+ \to B^+$ for $\langle B; \Lambda \rangle$ such that $d(\varphi(\underline{x})) \equiv d(\underline{x}) \,(\mathrm{mod} k)$ for all $\underline{x} \in B^+$.

We define a set of (n, m)–defining relations $\Delta \subseteq F(B) \times F(B)$ by

$$\Delta = \Big\{ (u, v) \in F(B) \times F(B) \,\Big|\, u = (i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}),\ v = (i, u_1^\lambda b_1^l u_{\lambda+r+1}^{m+sk}),$$
$$a_1^r, b_1^l \in B^+ \text{ and } a_1^r = b_1^l \text{ in } \langle B; \Lambda \rangle,$$
$$u_\alpha \in F(B), \alpha \in \{1, \ldots, \lambda\} \cup \{\lambda + r + 1, \ldots, m + sk\},$$
$$0 \leq \lambda \leq m + sk - r, 1 \leq r \leq m + sk,\ i \in \mathbb{N}_m,\ s \geq 1 \Big\}.$$

Thus, we get a (n, m)–semigroup presentation $\langle B; \Delta \rangle$ which is said to be induced by the semigroup presentation $\langle B; \Lambda \rangle$.
Our aim is to construct a good reduction for $\langle B; \Delta \rangle$. For that purpose, we will use the mapping ψ_0 and the fact that there exists an invariant $\rho : B^+ \to \mathbb{N}$ which is reduced by φ (since φ is a good reduction for $\langle B; \Lambda \rangle$). We will extend such invariant ρ on $F(B)$ and then, using φ we will define an auxiliary mapping $\psi' : F(B) \to F(B)$ which will reduce the extended invariant ρ. Afterwards, we will show and display the properties of the mapping ψ' as well as some properties of compositions of ψ_0 and ψ'. Applying these results and the properties of the reduction ψ_0 (see [3]), we will define an appropriate mapping $\psi : F(B) \to F(B)$ (by induction on hierarchy, combining ψ_0 and ψ'), and such ψ will be a good

reduction for $\langle B;\Delta\rangle$. Let us now proceed to this construction.

Being φ a good reduction for $\langle B;\Lambda\rangle$, there exists a mapping $\rho : B^+ \to \mathbb{N}$ such that $\varphi(\underline{x}) \neq \underline{x}$ implies $\rho(\varphi(\underline{x})) < \rho(\underline{x})$, for all $\underline{x} \in B^+$. We will extend the invariant ρ on $F(B)$ and define a mapping

$$\rho : F(B) \to \mathbb{N}_0, \text{ by induction on } \chi \text{ as follows:}$$

For $b \in B$, $\rho(b)$ is already defined and since the condition (I') for $\langle B;\Lambda\rangle$ implies that $\varphi(b) = b, b \in B$ (the elements from the basis are alone in the class), we can take $\rho(b) = 1, b \in B$ (the usual way of defining ρ on the basis in such cases).
Next, for $(i, a_1^{m+sk}) \in B_1\backslash B_0$, define $\rho(i, a_1^{m+sk}) = \rho(a_1^{m+sk})$; Assume that $\rho(v)$ is well defined for all $v \in B_p$ and extend the definition of ρ on B_p^* by induction on the dimension: We put $\rho(1) = 0$, assume that ρ is well defined for all $\underline{x} \in B_p^+$ with $d(\underline{x}) < q, (q \in \mathbb{N})$ and let $x_1^q \in B_p^+$.
If $x_1^q = x_1^\lambda a_1^r x_{\lambda+r+1}^q$ where: $a_1^r \in B^+, 1 \leq r \leq q$, $x_\lambda, x_{\lambda+r+1} \notin B$, $0 \leq \lambda \leq q-r$, and λ is the smallest such index, then $\rho(x_1^\lambda)$ and $\rho(x_{\lambda+r+1}^q)$ are well defined (by the hypothesis), and consequently define

$$\rho(x_1^q) = \rho(x_1^\lambda) + \rho(a_1^r) + \rho(x_{\lambda+r+1}^q).$$

If x_1^q doesn't satisfy the conditions above, we put $\rho(x_1^q) = \sum\limits_{j=1}^{q} \rho(x_j)$.

Now, for $u = (i, u_1^{m+sk}) \in B_{p+1}\backslash B_p$ we define $\rho(i, u_1^{m+sk}) = \rho(u_1^{m+sk})$.

Lemma 2.1. $\rho(i, x_1^\lambda a_1^r x_{\lambda+r+1}^{m+sk}) = \rho(x_1^\lambda) + \rho(a_1^r) + \rho(x_{\lambda+r+1}^{m+sk})$
where: $a_1^r \in B^+$, $1 \leq r \leq m+sk$, $x_\lambda, x_{\lambda+r+1} \notin B$, $0 \leq \lambda \leq m+sk-r$.

Proof. It is sufficient to show that $\rho(x_1^\lambda a_1^r x_{\lambda+r+1}^q) = \rho(x_1^\lambda) + \rho(a_1^r) + \rho(x_{\lambda+r+1}^q)$, where: $a_1^r \in B^+$, $x_\lambda, x_{\lambda+r+1} \notin B$, $0 \leq \lambda \leq m+sk-r$, $1 \leq r \leq m+sk$. Assume that the equality holds for all $\underline{x} \in F(B)^+$ satisfying the conditions above and such that $d(\underline{x}) < d(x_1^\lambda a_1^r x_{\lambda+r+1}^{m+sk}) = m+sk$. If λ is the smallest index such that $x_\lambda \notin B$, $a_1^r \in B^+$ and $x_{\lambda+r+1} \notin B$, the case is trivial (follows by definition). Hence, let $x_1^\lambda a_1^r x_{\lambda+r+1}^{m+sk} = x_1^j b_1^l x_{j+l+1}^\lambda a_1^r x_{\lambda+r+1}^{m+sk}$ where: $b_1^l \in B^+$, $x_j, x_{j+l+1} \notin B$, $0 \leq j \leq \lambda-l-1$, $(1 \leq l \leq \lambda-1)$ and let j be the smallest such index. Then $\rho(x_1^\lambda a_1^r x_{\lambda+r+1}^{m+sk}) = \rho(x_1^j b_1^l x_{j+l+1}^\lambda a_1^r x_{\lambda+r+1}^{m+sk}) = \rho(x_1^j) + \rho(b_1^l) + \rho(x_{j+l+1}^\lambda a_1^r x_{\lambda+r+1}^{m+sk})$ and $d(x_{j+l+1}^\lambda a_1^r x_{\lambda+r+1}^{m+sk}) = m+sk-j-l < m+sk$. Thus, $\rho(x_1^\lambda a_1^r x_{\lambda+r+1}^{m+sk}) = \rho(x_1^j) + \rho(b_1^l) + \rho(x_{j+l+1}^\lambda) + \rho(a_1^r) + \rho(x_{\lambda+r+1}^{m+sk}) = \rho(x_1^\lambda) + \rho(a_1^r) + \rho(x_{\lambda+r+1}^{m+sk})$. □

Define a mapping $\psi' : F(B) \to F(B)$ by induction on ρ as follows:

$$\psi'(b) = b, b \in B;$$

Let $u = (i, u_1^{m+sk}) \in F(B)\backslash B$, assume that ψ' is well defined for all $v \in F(B)$ such that $\rho(v) < \rho(u)$ and moreover, assume that

$$\psi'(v) \neq v \text{ implies } \rho(\psi'(v)) < \rho(v).$$

Let $u_1^{m+sk} = u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}$ where: $a_1^r \in B^+$, $1 \leq r \leq m+sk$, $\varphi(a_1^r) \neq a_1^r$, $u_\lambda, u_{\lambda+r+1} \notin B$, $0 \leq \lambda \leq m+sk-r$ and let λ (if exists) be the smallest such index. Then, $d(\varphi(a_1^r)) > m$ (by (I')), $d(\varphi(a_1^r)) \equiv d(a_1^r)\,(\mathrm{mod} k)$ (by (II')), and thus

$(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) \in F(B)$. Also, $\varphi(a_1^r) \neq a_1^r$ implies $\rho(\varphi(a_1^r)) < \rho(a_1^r)$, which implies that $\rho(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) < \rho(u)$. Consequently, define

$$\psi'(u) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}).$$

Now, $\rho(\psi'(u)) = \rho(\psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})) \leq \rho(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) < \rho(u)$ i.e. the mapping ψ' is well defined in this case, and it reduces the invariant ρ.
If u_1^{m+sk} doesn't satisfy the conditions above, we put

$$\psi'(u) = u.$$

Remark 2.1: Note that ψ' does not change the first coordinate of $(i, \underline{x}) \in F(B) \backslash B$. Furthermore, $\chi(\psi'(u)) = \chi(u)$, $u \in F(B)$. (Easy to verify, by induction on ρ).

Lemma 2.2. (1) *$\rho(\psi'(u)) \leq \rho(u)$ and $\rho(\psi'(u)) = \rho(u) \Leftrightarrow \psi'(u) = u$*

(2) *$\psi'(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$, where $(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) \in F(B)$ and $a_1^r \in B^+, 1 \leq r \leq m+sk$, $0 \leq \lambda \leq m+sk-r$.*

(3) *Let $(j, y_1^{m+sk}), (i, \underline{x}\, y_1^{m+sk} \underline{z}) \in F(B)$.
If $\psi'(j, y_1^{m+sk}) = (j, {y'}_1^{m+lk})$ then $\psi'(i, \underline{x}\, y_1^{m+sk} \underline{z}) = \psi'(i, \underline{x}\, {y'}_1^{m+lk} \underline{z})$.*

Proof. (1). Consequence from the fact that $\psi'(u) \neq u$ implies $\rho(\psi'(u)) < \rho(u)$ which is shown above, while defining ψ'.
(2). Firstly, will show that $\psi'(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$ for all $(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) \in F(B)$ where $a_1^r \in B^+$ $(1 \leq r \leq m+sk)$ and $u_\lambda, u_{\lambda+r+1} \notin B$, $(0 \leq \lambda \leq m+sk-r)$. If $\varphi(a_1^r) = a_1^r$ the case is trivial, so let $\varphi(a_1^r) \neq a_1^r$ and assume that the equality stands for all $v \in F(B)$ with $\rho(v) < \rho(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk})$. If λ is the smallest index such that $u_\lambda, u_{\lambda+r+1} \notin B$, the conclusion follows by definition. Let $u_1^\lambda = u_1^j b_1^l u_{j+l+1}^\lambda$ where: $b_1^l \in B^+$, $1 \leq l \leq \lambda - 1$, $\varphi(b_1^l) \neq b_1^l$, $u_j, u_{j+l+1} \notin B$, $0 \leq j \leq \lambda - l - 1$, and assume that j is the smallest such index. Then

$$\psi'(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^j b_1^l u_{j+l+1}^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^j \varphi(b_1^l) u_{j+l+1}^\lambda a_1^r u_{\lambda+r+1}^{m+sk}),$$

and moreover, $\varphi(b_1^l) \neq b_1^l$ implies $\rho(\varphi(b_1^l)) < \rho(b_1^l)$, which implies that $\rho(u_1^j \varphi(b_1^l) u_{j+l+1}^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) < \rho(u_1^j b_1^l u_{j+l+1}^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \rho(u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk})$. Thus,

$$\psi'(i, u_1^j \varphi(b_1^l) u_{j+l+1}^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^j \varphi(b_1^l) u_{j+l+1}^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) =$$
$$\psi'(i, u_1^j b_1^l u_{j+l+1}^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}).$$

To show the proposition (2), first assume that there exist $\mu' \in \{1, \ldots, \lambda\}$ and $\mu'' \in \{\lambda+r+1, \ldots, m+sk\}$ such that $u_{\mu'}, u_{\mu''} \notin B$. Moreover, let μ' and μ'' be the biggest and the smallest such index respectively. Then, $u_1^\lambda = u_1^{\mu'} b_1^l$ where $b_1^l \in B^+$, $l = \lambda - \mu' \geq 0$, and, $u_{\lambda+r+1}^{m+sk} = c_1^q u_{\mu''}^{m+sk}$ where $c_1^q \in B^+$, $q = \mu'' - \lambda - r - 1 \geq 0$. Now, using the equality from above, we get

$$\psi'(i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^{\mu'} b_1^l a_1^r c_1^q u_{\mu''}^{m+sk}) = \psi'(i, u_1^{\mu'} \varphi(b_1^l a_1^r c_1^q) u_{\mu''}^{m+sk}) =$$
$$\psi'(i, u_1^{\mu'} \varphi(b_1^l \varphi(a_1^r) c_1^q) u_{\mu''}^{m+sk}) = \psi'(i, u_1^{\mu'} b_1^l \varphi(a_1^r) c_1^q u_{\mu''}^{m+sk}) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}).$$

If $u_1^\lambda \in B^+$ or $u_{\lambda+r+1}^{m+sk} \in B^+$ the proof is analogical. If $u_1^{m+sk} \in B^+$ the statement follows from definition of ψ' and the properties of the reduction φ.

(3). By induction on ρ. Assume that the proposition stands for all $v \in F(B)$ with $\rho(v) < \rho(j, y_1^{m+sk})$ and let $\psi'(j, y_1^{m+sk}) = (j, {y'}_1^{m+lk}) \neq (j, y_1^{m+sk})$. Then $y_1^{m+sk} = y_1^\lambda a_1^r y_{\lambda+r+1}^{m+sk}$ for some $1 \leq r \leq m+sk$ and $0 \leq \lambda \leq m+sk-r$, where $a_1^r \in B^+$, $\varphi(a_1^r) \neq a_1^r$ and $y_\lambda, y_{\lambda+r+1} \notin B$. We can (but not need to) take λ to be the smallest such index. Now, $\psi'(j, y_1^\lambda \varphi(a_1^r) y_{\lambda+r+1}^{m+sk}) = (j, {y'}_1^{m+lk})$ and we have that $\rho(j, y_1^\lambda \varphi(a_1^r) y_{\lambda+r+1}^{m+sk}) < \rho(j, y_1^{m+sk})$. Applying (2) and the inductive hypothesis for the element $(j, y_1^\lambda \varphi(a_1^r) y_{\lambda+r+1}^{m+sk})$, we obtain that $\psi'(i, \underline{x}\, y_1^{m+sk} \underline{z}) = \psi'(i, \underline{x}\, y_1^\lambda a_1^r y_{\lambda+r+1}^{m+sk} \underline{z}) = \psi'(i, \underline{x}\, y_1^\lambda \varphi(a_1^r) y_{\lambda+r+1}^{m+sk} \underline{z}) = \psi'(i, \underline{x}\, {y'}_1^{m+lk} \underline{z})$. □

Lemma 2.3. (1) *If* $(u,v) \in \Delta$ *then* $\psi'(u) = \psi'(v)$.

(2) *If* $\psi'(u) \neq u$, *there exist a sequence* $u_0, u_1, \dots, u_{t-1}, u_t \in F(B)$ *such that* $u = u_0 \Delta u_1 \Delta \dots \Delta u_{t-1} \Delta u_t = \psi'(u)$, $(t \geq 1)$.

(3) $u \overline{\Delta} \psi'(u)$, $u \in F(B)$

(4) $\psi'(\psi'(u)) = \psi'(u)$, $u \in F(B)$.

Proof. (1). Consequence from (2) in Lemma 2.2.
(2). Let $u \in F(B)$ and let $\psi'(u) \neq u$. Then $u = (i, u_1^{m+sk}) \in F(B) \backslash B$ and assume that the proposition stands for all $v \in F(B)$ with $\rho(v) < \rho(u)$. Since $\psi'(u) \neq u$ we have that $\psi'(u) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$ for some $a_1^r \in B^+$ and $u_\lambda, u_{\lambda+r+1} \notin B$ where $\varphi(a_1^r) \neq a_1^r$, which implies that $\rho(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) < \rho(u)$. Also, $u = (i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}) \Delta (i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$.
If $\psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) = (i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$ we immediately get $u \Delta \psi'(u)$.
If $\psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) \neq (i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$, the hypothesis implies that there exists a sequence $u_0, u_1, \dots, u_{t-1}, u_t \in F(B)$, $(t \geq 1)$ such that

$$(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) \Delta u_1 \Delta \dots \Delta u_{t-1} \Delta \psi'(i,\, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}), \text{ and thus}$$
$$u \Delta (i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) \Delta u_1 \Delta \dots \Delta u_{t-1} \Delta \psi'(i,\, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) = \psi'(u).$$

(3). Direct consequence from (2).
(4). By induction on ρ. Clearly it holds on B, let $(i, u_1^{m+sk}) \in F(B) \backslash B$, and assume that $\psi'(\psi'(v)) = \psi'(v)$ for all $v \in F(B)$ with $\rho(v) < \rho(i, u_1^{m+sk})$. Let also $\psi'(i, u_1^{m+sk}) \neq (i, u_1^{m+sk})$. (Otherwise the equality is trivial). Then $\psi'(i, u_1^{m+sk}) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})$ for some $a_1^r \in B^+$ such that $\varphi(a_1^r) \neq a_1^r$ $(1 \leq r \leq m+sk)$, and some $u_\lambda, u_{\lambda+r+1}$ such that $u_\lambda, u_{\lambda+r+1} \notin B$, $(0 \leq \lambda \leq m+sk-r)$. Moreover, $\rho(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) < \rho(i, u_1^{m+sk})$ and by the hypothesis we get $\psi'(\psi'(i, u_1^{m+sk})) = \psi'(\psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk})) = \psi'(i, u_1^\lambda \varphi(a_1^r) u_{\lambda+r+1}^{m+sk}) = \psi'(i, u_1^{m+sk})$. □

Lemma 2.4. (1) $\psi_0 \psi' \psi_0(u) = \psi' \psi_0(u)$,

(2) $\psi' \psi_0 \psi'(u) = \psi' \psi_0(u)$, $u \in F(B)$.

Proof. (1). We will show that $v \in \psi_0(F(B))$ implies $\psi'(v) \in \psi_0(F(B))$. (By induction on ρ). Since it holds on B, let $v = (i, v_1^{m+sk}) \in \psi_0(F(B)) \backslash B$ and assume that the statement holds for all $z \in \psi_0(F(B))$ with $\rho(z) < \rho(i, v_1^{m+sk})$. Let also $\psi'(i, v_1^{m+sk}) \neq (i, v_1^{m+sk})$. Then, $(i, v_1^{m+sk}) = (i, v_1^{\lambda} a_1^r v_{\lambda+r+1}^{m+sk})$ for some $1 \leq r \leq m+sk$, $0 \leq \lambda \leq m+sk-r$, where: $a_1^r \in B^+$, $\varphi(a_1^r) \neq a_1^r$, $v_\lambda, v_{\lambda+r+1} \notin B$, $\rho(i, v_1^{\lambda}\varphi(a_1^r)v_{\lambda+r+1}^{m+sk}) < \rho(v)$, and, $\psi'(v) = \psi'(i, v_1^{m+sk}) = \psi'(i, v_1^{\lambda}\varphi(a_1^r)v_{\lambda+r+1}^{m+sk})$. Being $a_1^r \in B^+$, $\varphi(a_1^r) \in B^+$ and $(i, v_1^{\lambda} a_1^r v_{\lambda+r+1}^{m+sk}) \in \psi_0(F(B))$, it is easy to conclude that $(i, v_1^{\lambda}\varphi(a_1^r)v_{\lambda+r+1}^{m+sk}) \in \psi_0(F(B))$. Thus, and by the inductive hypothesis, we get that $\psi'(i, v_1^{\lambda}\varphi(a_1^r)v_{\lambda+r+1}^{m+sk}) \in \psi_0(F(B))$. Therefore, $\psi'(v) \in \psi_0(F(B))$. Consequently, $\psi'\psi_0(u) \in \psi_0(F(B))$ for all $u \in F(B)$, and now (1) follows from $\psi_0\psi_0 = \psi_0$ (Proposition 1.2).

(2). Let $u \in F(B)$ and let $\psi'(u) \neq u$. (Otherwise the case is trivial). Then $u = (i, u_1^{m+sk}) \in F(B)\backslash B$ and u_1^{m+sk} consists a subsequence $u_\lambda a_1^r u_{\lambda+r+1}$ such that $a_1^r \in B^+$ $(1 \leq r \leq m+sk)$, $\varphi(a_1^r) \neq a_1^r$, $u_\lambda, u_{\lambda+r+1} \notin B$, $(0 \leq \lambda \leq m+sk-r)$ and, $\psi'(i, u_1^{m+sk}) = \psi'(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk})$. Also, $\rho(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk}) < \rho(u)$. Assuming that the equality stands for all $v \in F(B)$ with $\rho(v) < \rho(u)$ we get

$$\psi'\psi_0\psi'(u) = \psi'\psi_0\psi'(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk}) = \psi'\psi_0(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk}).$$

Consider the images $\psi_0(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk})$ and $\psi_0(i, u_1^{\lambda}a_1^r u_{\lambda+r+1}^{m+sk})$. Recall Remark 1.1 and assume that $\psi_0(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk}) = (i, {u'}_1^{m+s_0k})$. Since $\varphi(a_1^r) \in B^+$, there exists an integer $\lambda_0 \geq \lambda$ such that $u'_{\lambda_0+1} \dots u'_{\lambda_0+d(\varphi(a_1^r))} = \varphi(a_1^r)$. (Easy to conclude, by induction on the length). Similarly, and being $a_1^r \in B^+$, we obtain that $\psi_0(i, u_1^{\lambda}a_1^r u_{\lambda+r+1}^{m+sk}) = \big(i, u'_1 \dots u'_{\lambda_0} a_1^r u'_{\lambda_0+d(\varphi(a_1^r))+1} \dots u'_{m+s_0k}\big)$. Therefore,

$$\begin{aligned}&\psi'\psi_0(i, u_1^{\lambda}\varphi(a_1^r)u_{\lambda+r+1}^{m+sk}) = \psi'\big(i, u'_1 \dots u'_{\lambda_0}\varphi(a_1^r)u'_{\lambda_0+d(\varphi(a_1^r))+1} \dots u'_{m+s_0k}\big) = \\ &\psi'\big(i, u'_1 \dots u'_{\lambda_0} a_1^r u'_{\lambda_0+d(\varphi(a_1^r))+1} \dots u'_{m+s_0k}\big) = \psi'\psi_0(i, u_1^{\lambda}a_1^r u_{\lambda+r+1}^{m+sk}) = \psi'\psi_0(u).\end{aligned}$$

Hence, $\psi'\psi_0\psi'(u) = \psi'\psi_0(u)$. □

Define a mapping $\psi : F(B) \to F(B)$ by induction on χ as follows:

$$\psi(b) = b, b \in B;$$

Let $u = (i, u_1^{m+sk}) \in F(B)\backslash B$ and assume that $\psi(v)$ is well defined for all $v \in F(B)$ such that $\chi(v) < \chi(u)$. Hence, $\psi(u_\mu)$ is well defined for all $\mu \in \mathbb{N}_{m+sk}$, and consequently define $\psi(u)$ by

$$\psi(i, u_1^{m+sk}) = \psi'\psi_0(i, \psi(u_1) \dots \psi(u_{m+sk})).$$

Lemma 2.5. (1) $\chi(\psi(u)) \leq \chi(u)$
(2) $\psi'(\psi(u)) = \psi(u)$
(3) $\psi_0(\psi(u)) = \psi(u)$
(4) $\psi(\psi(u)) = \psi(u)$, *for all* $u \in F(B)$.

Proof. (1). By induction on the hierarchy. $\chi(\psi(b)) = \chi(b), b \in B$, assume that $\chi(\psi(v)) \leq \chi(v)$ for all $v \in B_p$ and let $u = (i, u_1^{m+sk}) \in B_{p+1}\backslash B_p$. Then,

$\chi(\psi(u_\alpha)) \leq \chi(u_\alpha)$, $\alpha \in \mathbb{N}_{m+sk}$ and applying the properties of χ for the mappings ψ' and ψ_0 respectively (Remark 2.1 and Remark 1.1), we get

$$\chi(\psi(i, u_1^{m+sk})) = \chi(\psi'\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk}))) =$$
$$\chi(\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk}))) \leq \chi(i, \psi(u_1)\dots\psi(u_{m+sk})) \leq \chi(i, u_1^{m+sk}).$$

(2). Consequence from (4) in Lemma 2.3.

(3). Consequence from (1) in Lemma 2.4.

(4). Firstly, we will show that all $u \in F(B)\backslash B$ satisfy

$\psi(u) = (i, w_1^{m+rk})$ where $i \in \mathbb{N}_m$ and $w_\eta \in \psi(F(B))$, $\eta \in \mathbb{N}_{m+rk}$.

By induction on χ. For $u \in B_1\backslash B$, $u = (i, a_1^{m+sk})$ where $i \in \mathbb{N}_m$ and $a_1^{m+sk} \in B^+$. Hence, $\psi(u) = \psi'\psi_0(i, a_1^{m+sk}) = \psi'(i, a_1^{m+sk}) = (i, \varphi(a_1^{m+sk}))$ and the conclusion follows immediately, being $B \subseteq \psi(F(B))$. Assume that the statement holds for all $u' \in B_p$ and let $u \in B_{p+1}\backslash B_p$. Then, $u = (i, u_1^{m+sk})$ for some $i \in \mathbb{N}_m$ and $u_1^{m+sk} \in B_p^+$, and $\psi(u) = \psi(i, u_1^{m+sk}) = \psi'\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk}))$. Recalling that none of the mappings ψ_0 and ψ' changes the first coordinate, assume that $\psi'\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk})) = (i, w_1^{m+rk})$. Let also $\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk})) = (i, v_1^{m+lk})$. We will show that $v_\mu \in \psi(F(B))$, $\mu \in \mathbb{N}_{m+lk}$:
If $\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk})) = (i, \psi(u_1)\dots\psi(u_{m+sk}))$ the conclusion is trivial. If $\psi_0(i, \psi(u_1)\dots\psi(u_{m+sk})) \neq (i, \psi(u_1)\dots\psi(u_{m+sk}))$, then there exists $\vartheta \in \mathbb{N}_{sk}^0$ such that $\psi(u_{\vartheta+\beta}) = (\beta, y_1^{m+qk}), \beta \in \mathbb{N}_m$ (since $\psi_0\psi = \psi$), and thus $(i, v_1^{m+lk}) = \psi_0(i, \psi(u_1)\dots\psi(u_\vartheta)y_1^{m+qk}\psi(u_{\vartheta+m+1})\dots\psi(u_{m+sk}))$. Moreover, the hypothesis implies that $y_j \in \psi(F(B)), j \in \mathbb{N}_{m+qk}$. Consequently, if

$\psi_0(i, \psi(u_1)\dots\psi(u_\vartheta)y_1^{m+qk}\psi(u_{\vartheta+m+1})\dots\psi(u_{m+sk})) =$
$(i, \psi(u_1)\dots\psi(u_\vartheta)y_1^{m+qk}\psi(u_{\vartheta+m+1})\dots\psi(u_{m+sk}))$,

we immediately get $v_\mu \in \psi(F(B))$, $\mu \in \mathbb{N}_{m+lk}$. Otherwise, the conclusion follows by induction (i.e. repeating the same process).
Hence, we have that $\psi(u) = (i, w_1^{m+rk}) = \psi'(i, v_1^{m+lk})$, where $v_\mu \in \psi(F(B))$, $\mu \in \mathbb{N}_{m+lk}$. Thus, and being $B \subseteq \psi(F(B))$ it is easy to verify that $w_\eta \in \psi(F(B))$, $\eta \in \mathbb{N}_{m+rk}$. (Induction on ρ).

Let us now proof the statement (4). By induction on χ.
It is clear that $\psi\psi(b) = \psi(b), b \in B$, assume that $\psi\psi(z) = \psi(z)$ for all $z \in F(B)$ with $\chi(z) \leq p$, and let $(i, u_1^{m+sk}) \in B_{p+1}\backslash B_p$. We have showed that $\psi(i, u_1^{m+sk}) = (i, w_1^{m+rk})$ (for some $r \geq 1$) where $w_\eta \in \psi(F(B))$, $\eta \in \mathbb{N}_{m+rk}$ and thus $w_\eta = \psi(w'_\eta)$ for some $w'_\eta \in F(B)$, $\eta \in \mathbb{N}_{m+rk}$. Furthermore, $\chi(w'_\eta) < \chi(i, u_1^{m+sk})$, $\eta \in \mathbb{N}_{m+rk}$ and applying the hypothesis we get that $\psi\psi(w'_\eta) = \psi(w'_\eta)$, i.e. $\psi(w_\eta) = w_\eta$ for all $\eta \in \mathbb{N}_{m+rk}$. Thus, and by (2) and (3) (this lemma) we obtain

$$\psi\psi(i, u_1^{m+sk}) = \psi(i, w_1^{m+rk}) = \psi'\psi_0(i, \psi(w_1)\dots\psi(w_{m+rk})) =$$
$$\psi'\psi_0(i, w_1^{m+rk}) = \psi'\psi_0\psi(i, u_1^{m+sk}) = \psi(i, u_1^{m+sk}). \qquad \square$$

Proposition 2.1. *The mapping ψ is a good reduction for $\langle B; \Delta\rangle$.*

Proof. (i). Let $u\Delta v$, i.e. let $u = (i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk})$, $v = (i, u_1^\lambda b_1^l u_{\lambda+r+1}^{m+sk})$ where $a_1^r = b_1^l$ in $\langle B; \Lambda\rangle$, $0 \leq \lambda \leq m + sk - r$, $1 \leq r \leq m + sk$. Then, $\varphi(a_1^r) = \varphi(b_1^l)$ and by Lemma 2.4-(2) and Lemma 2.2-(2) we obtain

$$\psi(u)=\psi'\psi_0(i,\psi(u_1)\dots\psi(u_\lambda)a_1^r\psi(u_{\lambda+r+1})\dots\psi(u_{m+sk}))=$$
$$\psi'\psi_0\psi'(i,\psi(u_1)\dots\psi(u_\lambda)a_1^r\psi(u_{\lambda+r+1})\dots\psi(u_{m+sk}))=$$
$$\psi'\psi_0\psi'(i,\psi(u_1)\dots\psi(u_\lambda)\varphi(a_1^r)\psi(u_{\lambda+r+1})\dots\psi(u_{m+sk}))=$$
$$\psi'\psi_0\psi'(i,\psi(u_1)\dots\psi(u_\lambda)\varphi(b_1^l)\psi(u_{\lambda+r+1})\dots\psi(u_{m+sk}))=$$
$$\psi'\psi_0\psi'(i,\psi(u_1)\dots\psi(u_\lambda)b_1^l\psi(u_{\lambda+r+1})\dots\psi(u_{m+sk}))=$$
$$\psi'\psi_0(i,\psi(u_1)\dots\psi(u_\lambda)b_1^l\psi(u_{\lambda+r+1})\dots\psi(u_{m+sk}))=\psi(v).$$

(ii). Let $(i,x'(1,y)\dots(m,y)x'')\in F(B)$. Then

$$\psi(i,x'(1,y)\dots(m,y)x'')=\psi'\psi_0(i,\underline{\psi(x')}\psi(1,y)\dots\psi(m,y)\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}\psi'\psi_0(1,\underline{\psi(y)})\dots\psi'\psi_0(m,\underline{\psi(y)})\underline{\psi(x'')}),$$

where $\underline{\psi(x')}$, $\underline{\psi(x'')}$ and $\underline{\psi(y)}$ denote the sequences of the images by ψ of the elements in the sequences x',x'' and y respectively.
Assume that $\psi_0(j,\underline{\psi(y)})=(j,\underline{y^0})$ and that $\psi'(j,\underline{y^0})=(j,\underline{y'}),j\in\mathbb{N}_m$. (Note that such assumptions are correct, according to Remark 1.1 and Remark 2.1). Applying (2) from Lemma 2.4, (3) from Lemma 2.2 and the properties of ψ_0, we get that

$$\psi'\psi_0(i,\underline{\psi(x')}\psi'\psi_0(1,\underline{\psi(y)})\dots\psi'\psi_0(m,\underline{\psi(y)})\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}(1,\underline{y'})\dots(m,\underline{y'})\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}\,\underline{y'}\,\underline{\psi(x'')})=$$
$$\psi'\psi_0\psi'(i,\underline{\psi(x')}\,\underline{y'}\,\underline{\psi(x'')})=$$
$$\psi'\psi_0\psi'(i,\underline{\psi(x')}\,\underline{y^0}\,\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}\,\underline{y^0}\,\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}(1,\underline{y^0})\dots(m,\underline{y^0})\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}(1,\underline{\psi(y)})\dots(m,\underline{\psi(y)})\underline{\psi(x'')})=$$
$$\psi'\psi_0(i,\underline{\psi(x')}\,\underline{\psi(y)}\,\underline{\psi(x'')})=\psi(i,x'yx'').$$

(iii). Follows from (4) in Lemma 2.5, applying the definition of ψ.

(iv). By induction on χ . If $u\in B$ then $\psi(u)=u\overline{\Delta}u$. Assume that $\psi(v)\overline{\Delta}v$ for all $v\in F(B)$ with $\chi(v)\leq p$, and let $u=(i,u_1^{m+sk})\in B_{p+1}\backslash B_p$. Applying the hypothesis and the corresponding property for ψ' and ψ_0 respectively, we get

$$\psi(i,u_1^{m+sk})=\psi'\psi_0(i,\psi(u_1)\dots\psi(u_{m+sk}))\overline{\Delta}\,\psi_0(i,\psi(u_1)\dots\psi(u_{m+sk}))\approx$$
$$(i,\psi(u_1)\dots\psi(u_{m+sk}))=f_i(\psi(u_1)\dots\psi(u_{m+sk}))\overline{\Delta}\,f_i(u_1^{m+sk})=(i,u_1^{m+sk}).$$

(v). Shown in Lemma 2.5.

Hence, the conditions (i)-(v) are satisfied and thus ψ is a reduction for $\langle B;\Delta\rangle$. Moreover, for a given $u\in F(B)$, the reduced represent $\psi(u)$ can be determined in a finite number of steps - according to its definition and since it can be done so for the corresponding images of the mappings ψ_0 and ψ' respectively. Recall that ψ_0 reduces the length (Proposition 1.2) and that ψ' reduces the invariant ρ (Lemma 2.2-(1)). Therefore, ψ is a good reduction for $\langle B;\Delta\rangle$. An element u from $F(B)$ is reduced if and only if $u \in B$ or $u = (i, u_1^{m+sk})$ where: $\psi(u_\alpha) = u_\alpha$, $\alpha \in \mathbb{N}_{m+sk}$; there is no $\mu \in \mathbb{N}_{sk}^0$ such that $u_{\mu+\beta} = (\beta, w_1^{m+rk})$ for each $\beta \in \mathbb{N}_m$; and the sequence u_1^{m+sk} doesn't consist a subsequence $u_\lambda a_1^r u_{\lambda+r+1}$ such that $a_1^r \in B^+$, $\varphi(a_1^r) \neq a_1^r$ and $u_\lambda, u_{\lambda+r+1} \notin B$, where $1 \leq r \leq m+sk$, $0 \leq \lambda \leq m+sk-r$. □

Consider now a semigroup presentation $\langle B; \Lambda \rangle$ satisfying the conditions:

(I) $d(\underline{x}), d(\underline{z}) > m$ and $d(\underline{x}) \equiv d(\underline{z}) \,(\mathrm{mod} k)$ for all $(\underline{x}, \underline{z}) \in \Lambda$

(II) There exists a good reduction $\varphi : B^+ \to B^+$ for $\langle B; \Lambda \rangle$.

In this case, $d(\varphi(\underline{x})) \equiv d(\underline{x}) \,(\mathrm{mod} k)$ for all $\underline{x} \in B^+$, since the dimensions of the elements in the same class are equivalent modulo k. (Easy to show, applying condition (I)). Thus, $\langle B; \Lambda \rangle$ satisfies the conditions (I') & (II') given at the beginning. The converse is also true, i.e. if a semigroup presentation $\langle B; \Lambda \rangle$ satisfies the conditions (I') & (II'), then $d(\underline{x}) \equiv d(\underline{z}) \,(\mathrm{mod} k)$ for all $(\underline{x}, \underline{z}) \in \Lambda$ (since $d(\underline{x}) \equiv d(\varphi(\underline{x})) = d(\varphi(\underline{z}) \equiv d(\underline{z}) \,(\mathrm{mod} k))$. Hence, (I) & (II) $\iff$ (I') & (II').

Our main result follows.

Theorem 2.1. *Let $\langle B; \Lambda \rangle$ be a presentation of a binary semigroup satisfying:*

(I) $d(\underline{x}), d(\underline{z}) > m$ *and* $d(\underline{x}) \equiv d(\underline{z}) \,(\mathrm{mod} k)$ *for all* $(\underline{x}, \underline{z}) \in \Lambda$

(II) *There exists a good reduction φ for $\langle B; \Lambda \rangle$.*

Let $\Delta \subseteq F(B) \times F(B)$ be the following set of $(m+k,m)$–defining relations

$$\Delta = \Big\{ (u,v) \in F(B) \times F(B) \,\Big|\, u = (i, u_1^\lambda a_1^r u_{\lambda+r+1}^{m+sk}),\ v = (i, u_1^\lambda b_1^l u_{\lambda+r+1}^{m+sk}),$$
$$a_1^r, b_1^l \in B^+ \text{ and } a_1^r = b_1^l \text{ in } \langle B; \Lambda \rangle,$$
$$u_\alpha \in F(B), \alpha \in \{1, \dots, \lambda\} \cup \{\lambda+r+1, \dots, m+sk\},$$
$$0 \leq \lambda \leq m+sk-r, 1 \leq r \leq m+sk,\ i \in \mathbb{N}_m,\ s \geq 1 \Big\}.$$

Then a good reduction ψ for the $(m+k,m)$–semigroup presentation $\langle B; \Delta \rangle$ can be constructed. (Essentially ψ is induced by φ). □

Corollary 2.1.1. *There exists a good (satisfactory) description of the corresponding $(m+k,m)$–semigroup with presentation $\langle B; \Delta \rangle$.*

Proof. Since ψ is a good (effective) reduction for $\langle B; \Delta \rangle$, the statement follows from Theorem 1.1 and from the fact that $\psi(u), u \in F(B)$ can be calculated in a finite number of steps. (See also [4], p.149–150). □

Corollary 2.1.2. *There exists an explicit description of the congruence $\overline{\Delta}$.*

Proof. Define a sequence $\Delta_0, \Delta_1, \dots, \Delta_p, \dots$ of $(m+k,m)$–relations on the sets $B_0, B_1, \dots, B_p, \dots$ respectively, by induction, as follows:

$$\Delta_0 = \{(b,b) \,|\, b \in B\};$$

$$\Delta_1 = \Delta_0 \bigcup \Big\{ \big((i, a_1^{m+sk}), (i, b_1^{m+qk})\big) \,\Big|\, i \in \mathbb{N}_m, s, q \geq 1, a_1^{m+sk} = b_1^{m+qk} \text{ in } \langle B; \Lambda \rangle \Big\};$$

Assume that Δ_p is defined on B_p and define Δ_{p+1} on B_{p+1} by

$$\Delta_{p+1} = \Delta_p \bigcup \Big\{ \big((i, u_1^{m+sk}), (i, v_1^{m+sk})\big) \Big| \, i \in \mathbb{N}_m, s \geq 1, u_\eta \Delta_p v_\eta, \eta \in \mathbb{N}_{m+sk} \Big\}$$
$$\bigcup \Big\{ \big((i, u_1^\alpha a_1^r u_{\alpha+1}^{m+sk-r}), (i, v_1^\alpha b_1^l v_{\alpha+1}^{m+sk-r})\big) \Big| \, i \in \mathbb{N}_m, s \geq 1,$$
$$a_1^r = b_1^l \text{ in } \langle B; \Lambda \rangle, u_\eta \Delta_p v_\eta, \eta \in \mathbb{N}_{m+sk-r}, \alpha \in \mathbb{N}_{m+sk-r}^0 \Big\}$$
$$\bigcup \Big\{ \big((i, u_1^\alpha (1, \underline{y}) \dots (m, \underline{y}) u_{\alpha+1}^{sk}), (i, v_1^\alpha \underline{y} v_{\alpha+1}^{sk})\big) \Big| \, i \in \mathbb{N}_m,$$
$$s \geq 1, (j, \underline{y}) \in B_p, u_\eta \Delta_p v_\eta, \eta \in \mathbb{N}_{sk}, 0 \leq \alpha \leq sk \Big\}$$
$$\bigcup \Big\{ \big((i, u_1^\alpha \underline{y} u_{\alpha+1}^{sk}), (i, v_1^\alpha (1, \underline{y}) \dots (m, \underline{y}) v_{\alpha+1}^{sk})\big) \Big| \, i \in \mathbb{N}_m,$$
$$s \geq 1, (j, \underline{y}) \in B_p, u_\eta \Delta_p v_\eta, \eta \in \mathbb{N}_{sk}, 0 \leq \alpha \leq sk \Big\}.$$

Let $\Delta_* = \bigcup_{p \geq 0} \Delta_p$. Then, $\overline{\Delta}$ is the smallest transitive extension of Δ_*. (Easy to verify, being $\overline{\Delta} = \ker \psi$ and having the standard description of $\overline{\Delta}$, see [4],§1). □

At the end, we consider one special case of Theorem 2.1.
Let $\langle B; \Lambda' \rangle$ be a presentation of a semigroup such that $d(\underline{x}), d(\underline{z}) > m$ for all $(\underline{x}, \underline{z}) \in \Lambda'$, let φ be a good reduction for $\langle B; \Lambda' \rangle$ and let $k = 1$.
Then, the conditions (I) & (II) from Theorem 2.1 are satisfied. Consequently, $\langle B; \Lambda' \rangle$ induces an $(m+1, m)$–semigroup presentation $\langle B; \Delta' \rangle$ and φ induces a good reduction ψ for $\langle B; \Delta' \rangle$. The set of the corresponding $(m+1, m)$–defining relations Δ' is given by Theorem 2.1, taking $k = 1$.

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Institute of Mathematics, Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University, P.O.Box 162, 1000 Skopje, Republic of Macedonia
E-mail address: donco@iunona.pmf.ukim.edu.mk

Orce Nikolov 96, 1000 Skopje, Republic of Macedonia
E-mail address: irena.stojmenovska@gmail.com