Математички Билтен
29 (LV)
2005 (77-88)
Скопје, Македонија

ISSN 0351-336X

EXTENSION OF LINEAR n-FUNCTIONALS DOMINATED BY SEMI-n-NORM

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Abstract

In this paper we establish an extension theorem for a complex linear n-functional dominated by a semi-n-norm.

1. Introduction

Gähler's studies on 2-metric and 2-normed spaces [2-4] led to serious and interesting studies of these spaces. Continuing these studies several authors [1, 5, 6-9, 10, 11-12, 13-14] have recently obtained results on the stucture of higher dimensional normed spaces and have established Hahn-Banach type extension theorems.

If M and N are two linear subspaces of a real linear space E then it was shown by Gähler [4] that a linear 2-functional with domain $M \times N$ does not necessarily admit a linear extension to $E \times E$. However in 1969, White [13] showed that a 2-normed linear space E, a bounded linear 2-functional with domain Mx $[x]$ where $[x]$ is the linear space generated by $x \in E$ has an extension to Ex $[x]$. This result was generalised by Lal *et al.* The main result embodied in Theorem 2.2 is an extension of a linear n-functional dominated by a semi-n-norm.

2. Let E be a linear space over the field $\mathbf{K}(\mathbf{K} = \mathbf{R}.$ or $\mathbf{C})$

Definition 2.1 A mapping $\|.,\dots,.\|: E^n \to \mathbf{R}$ is called an n-norm on

E, if for all $x_i, y_1 \in E$, $i = 1, 2, \ldots, n$ and $\alpha \in \mathbf{K}$.

$$\|x_1, \ldots, x_n\| \geq o, \quad \text{and} \quad \|x_1, \ldots, x_n\| = o \quad \text{if and only if the set } \{x_1, \ldots, x_n\} \text{ is linearly dependent}; \tag{2.1}$$

$$\|x_1, \ldots, x_n\| = \|\pi(x_1), \ldots, \pi(x_n)\| \quad \text{for every bijection } \pi: \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}; \tag{2.2}$$

$$\|\alpha x_1, \ldots, x_n\| = |\alpha| \, \|x_1, \ldots, x_n\|; \tag{2.3}$$

$$\|x_1 + y_1, x_2, \ldots, x_n\| \leq \|x_1, x_2 \ldots, x_n\| + \|y_1, x_2, \ldots, x_n\|. \tag{2.4}$$

The pair $(E^n, \|., \ldots, .\|)$ is caled an n-normed linear space over $\mathbf{K}$.

Definition 2.2. A mapping p: $E^n \to \mathbf{R}$ is called a semi-n-norm on E if

$$p(x_1, \ldots, x_i + y_i, \ldots, x_n) \leq p(x_1, \ldots, x_i, \ldots, x_n) + p(x_1, \ldots, y_i, \ldots, x_n); \tag{2.5}$$

$$p(\lambda_1 x_1, \ldots, \lambda_n x_n) = |\lambda_1| \ldots |\lambda_n| p(x_1, \ldots, x_n); \tag{2.6}$$

for all $\lambda_i \in \mathbf{K}$ and $x_i, y_i \in E$, $i = 1, 2, \ldots, n$.

Remark 2.1. If the semi-n-norm p be such that $p(x_1, \ldots, x_i, \ldots, x_n) = 0$ if and only in the set $\{x_1, \ldots, x_n\}$ is linearly depedent then p is an n-norm.

To see this, we first prove that, for $i < j$,

$$\begin{aligned} &p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ &= p(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n). \end{aligned}$$

For,

$$\begin{aligned} &p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ &\leq p(x_1, \ldots, x_{i-1}, x_i + x_j, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ &+ p(x_1, \ldots, x_{i-1}, -x_j, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ &= p(x_1, \ldots, x_{i-1}, x_i + x_j, x_{i+1}, \ldots, x_{j-1}, x_j + x_i - x_i, x_{j+1}, \ldots, x_n) \\ &\leq p(x_1, \ldots, x_{i-1}, x_i + x_j, x_{i+1}, \ldots, x_{j-1}, x_j + x_i, x_{j+1}, \ldots, x_n) \\ &+ p(x_1, \ldots, x_{i-1}, x_i + x_j, x_{i+1}, \ldots, x_{j-1}, -x_i, x_{j+1}, \ldots, x_n) \\ &\leq p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, -x_i, x_{j+1}, \ldots, x_n) \\ &+ p(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, -x_i, x_{j+1}, \ldots, x_n) \\ &= p(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n). \end{aligned}$$

The reverse inequality can be similarly established and we get

$$\begin{aligned} &p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ &= p(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n) \end{aligned}$$

from which it follows that

$$p(x_1, x_2, \ldots, x_n) = p(\pi(x_1), \pi(x_2), \ldots, \pi(x_n))$$

for every bijection π: $\{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}$ and it follows that p is an n-norm on E.

Definition 2.3. Let X_i, $i = 1, 2, \ldots, n$ be n-linear subspaces of the linear space E over the field $\mathbf{K}$. A mapping f: $X_1 \times \ldots \times X_n \to \mathbf{K}$ is said to be a linear n-functional if for all $\alpha_i \in \mathbf{K}$ and $x_i, y_i \in X_i$, $i = 1, 2, \ldots, n$,

$$f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \sum_{\substack{z_i \in \{x_i, y_i\} \\ i=1,\ldots,n}} f(z_1, z_2, \ldots, z_n)\,; \tag{2.7}$$

$$f(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_1 \alpha_2 \ldots \alpha_n f(x_1, x_2, \ldots, x_n)\,. \tag{2.8}$$

We first establish the following

Theorem 2.1. Let E be a linear space over $\mathbf{K}$, $\dim E \geq n$ and p be a semi-n-norm on E. Let M be a subspace of E, $x_1, \ldots, x_{n-1}$ be $(n-1)$ non-zero elements of E and f be a linear n-functional on $M \times [x_1] \times \ldots \times [x_{n-1}]$ satisfying

$$|f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})| \leq p(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \tag{2.9}$$

for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in \mathbf{M} \times [x_1] \times \ldots \times [x_{n-1}]$. Then exists a linear n-functional F on $E \times [x_1] \times \ldots \times [x_{n-1}]$ such that

$$|F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})| \leq p(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$$

for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in E \times [x_1] \times \ldots \times [x_{n-1}]$ and

$$F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$$

for every

$$(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in \mathbf{M} \times [x_1] \times \ldots \times [x_{n-1}]\,.$$

Proof. Define a functional $\tilde{f}$ on $\mathbf{M}$ by $\tilde{f}(x) = f(x, x_1, \ldots, x_{n-1})$, for every $x \in \mathbf{M}$. Define a functional $\tilde{p}$ on E by $\tilde{p}(x) = p(x, x_1, \ldots, x_{n-1})$, for every $x \in E$. Then $\tilde{p}$ defines a semi-norm on E, and we have a linear functional $\tilde{f}$ on $\mathbf{M}$ such that $|\tilde{f}(x)| \leq \tilde{p}(x)$, for every $x \in \mathbf{M}$. Appealing to the Hahn-Banach theorem, we get a linear functional g on E such that

$$g(x) = \tilde{f}(x) \quad \text{for all} \quad x \in \mathbf{M} \quad \text{and} \tag{2.10}$$

$$|g(x)| \leq \tilde{p}(x) \quad \text{for all} \quad x \in E\,. \tag{2.11}$$

Define F on $E \times [x_1] \times \ldots \times [x_{n-1}]$ by

$$f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = \lambda_1 \ldots \lambda_{n-1}\; g(x)$$

for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in E \times [x_1] \times \ldots \times [x_{n-1}]$. Then F is a linear n-functional on $E \times [x_1] \times \ldots \times [x_{n-1}]$. Using (2.10) for all $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in \mathbf{M} \times [x_1] \times \ldots \times [x_{n-1}]$, we have

$$\begin{aligned} F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) &= \lambda_1 \ldots \lambda_{n-1} f(x, x_1, \ \ldots, x_{n-1}) \\ &= f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \,. \end{aligned}$$

for all $\lambda_1, \ldots, \lambda_{n-1} \in \mathbf{K}$ and for every

$$(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in \mathbf{M} \times [x_1] \times \ldots \times [x_{n-1}].$$

Furthemore, for all

$$(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in E \times [x_1] \times \ldots \times [x_{n-1}],$$

we have,

$$\begin{aligned} |F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) &= |\lambda_1| \ldots |\lambda_{n-1}|\, |g(x)| \\ &\leq |\lambda_1| \ldots |\lambda_{n-1}| \tilde{p}(x) \ \text{using} \quad (2.11) \\ &= |\lambda_1| \ldots |\lambda_{n-1}| p(x, x_1, \ldots, x_{n-1}) \\ &= p(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \end{aligned}$$

and this establishes the theorem.

Definition 2.4. Let $L_1, L_2, \ldots, L_{n-1}$ be $(n-1)$ finite dimensional subspaces of E and let

$$L_1 = [e_1^1, \ldots, e_{k_1}^1],\ L_2 = [e_1^2, \ldots, e_{k_2}^2], \ldots, L_{n-1} = [e_1^{n-1}, \ldots, e_{k_{n-1}}^{n-1}],$$

where $k_1, k_2, \ldots, k_{n-1}$ are dimensions of $L_1, L_2, \ldots, L_{n-1}$ respectively. Then p is said to possess property P_R along $L_1, L_2, \ldots, L_{n-1}$ if for $\lambda_{l_i}^i \in \mathbf{K}$, $l_i = 1, 2, \ldots, k_i$; $i = 1, 2, \ldots, n-1$

$$\begin{aligned} & p\left(x, \sum_{l_1=1}^{k_1} \lambda_{l_1}^1 e_{l_1}^1, \sum_{l_2=1}^{k_2} \lambda_{l_2}^2 e_{l_2}^2, \ldots, \sum_{l_{n-1}=1}^{k_{n-1}} \lambda_{l_{n-1}}^{n-1} e_{l_{n-1}}^{n-1}\right) \\ &= \sum_{l_1=1}^{k_1} \sum_{l_2=1}^{k_2} \cdots \sum_{l_{n-1}=1}^{k_{n-1}} p(x, \lambda_{l_1}^1 e_{l_1}^1, \lambda_{l_2}^2 e_{l_2}^2, \ldots, \lambda_{l_{n-1}}^{n-1} e_{l_{n-1}}^{n-1}) \,. \end{aligned} \tag{2.12}$$

Similarly p is said to possess property P_L along $L_1, L_2, \ldots, L_{n-1}$ if

$$\begin{aligned} p&\left(\sum_{l_1=1}^{k_1} \lambda_{l_1}^1 e_{l_1}^1, \ldots, \sum_{l_{n-1}=1}^{k_{n-1}} \lambda_{l_{n-1}}^{n-1} e_{l_{n-1}}^{n-1}, x\right) \\ &= \sum_{l_1=1}^{k_1} \cdots \sum_{l_{n-1}=1}^{k_{n-1}} p(\lambda_{l_1}^1 e_{l_1}^1, \ldots, \lambda_{l_{n-1}}^{n-1} e_{l_{n-1}}^{n-1}, x) \end{aligned} \tag{2.13}$$

for all $\lambda_{l_i}^i \in \mathbf{K}$, $l_i = 1, 2, \ldots, k_i$; $i = 1, 2, \ldots, n-1$.

We also note the following

Proposition 2.1. Let $L_1, L_2, \ldots, L_{n-1}$ be $(n-1)$ finite dimensional linear subspaces of E and let p be a semi-n-norm on E having property P_R along $L_1, \ldots, L_{n-1}$ independent of the choice of the generating elements of any of the subspaces $L_1, \ldots, L_{n-1}$ having dimension > 1. Then $p(x, x_1, \ldots, x_{n-1}) = 0$, for every $(x, x_1, \ldots, x_{n-1}) \in E \times L_1 \times \ldots \times L_{n-1}$.

Proof. Without loss of generality, we assume that p is a semi-n-norm on E, having property P_R along $L_1, L_2, \ldots, L_{n-1}$, independent of the choice of generating elements of L_1. Let $\dim L_1 = k_1$, $k_1 > 1$ and $L_1 = [e_1^1, \ldots, e_{k_1}^1]$. We show that $p(x, e_l^1, x_2, \ldots, x_{n-1}) = 0$ for every $x \in E$, $x_i \in L_i$, $i = 2, \ldots, n-1$ and for every $l = 1, \ldots, k_1$.

For any $l = 1, 2, \ldots, k_1$, choose $m \neq 1$ and $1 \leq k_1$. Then as

$$\{e_1^1, \ldots, e_{m-1}^1, e_m^1 + e_l^1, e_{m+1}^1, \ldots, e_{k_1}^1\}$$

is linearly independent and it also generates L_1, by our assumption p satisfies the property along $L_1, \ldots, L_{n-1}$ with $L_1 = [e_l^l, \ldots, e_{k_l}^l]$ and also with $L_1 = [e_1^1, \ldots, e_{m-1}^1, e_m^1 + e_l^1, e_{m+1}^1, \ldots, e_{k_1}^1]$. Hence, for any $x \in E$, $x_i \in L_i$, $i = 2, \ldots, n-1$, we have

$$\begin{aligned} p(x, e_m^l, x_2, \ldots, x_{n-1}) &= p((x, (e_m^l + e_l^l) + (-e_l^l), x_2, \ldots, x_{n-1}) \\ &= p(x, e_m^l + e_l^l, x_2, \ldots, x_{n-1}) + p(x, -e_l^l, x_2, \ldots, x_{n-1}) \\ &\quad \text{(as } p \text{ satisfies properti } P_R \text{ with } L_1 \text{ in second form)} \\ &= p(x, e_m^1, x_2, \ldots, x_{n-1}) + p(x, e_l^1, x_2, \ldots, x_{n-1}) + p(x, e_l^1, x_2, \ldots, x_{n-1}) \\ &\quad \text{(as } p \text{ satisfies property } P_R \text{ with } L_1 \text{ in first form and} \\ &\quad \text{as } p \text{ is a semi-}n\text{-norm).} \end{aligned}$$

and this proves that $p(x, e_l^l, x_2, \ldots, x_{n-1}) = 0$. The proposition now follows immedately.

Now we state and prove the following

Theorem 2.2. Let E be a linear space over $\mathbf{K}$, $\dim E \geq n$ and let $M, L_1, \ldots, L_{n-1}$ be n linear subspaces of E with

$$L_1 = [e_l^1, \ldots, e_{k_1}^1],\ L_2 = [e_l^2, \ldots, e_{k_2}^2], \ldots, L_{n-1} = [e_1^{n-1}, \ldots, e_{k_{n-1}}^{n-1}].$$

Let f be a linear n-functional on $M \times L_1 \times \ldots \times L_{n-1}$. If there exist a semi-$n$-norm p on E satisfying property P_R along $L_1, L_2, \ldots, L_{n-1}$ and

$$|f(x, x_1, \ldots, x_{n-1})| \leq p(x, x_1, \ldots, x_{n-1}) \tag{2.14}$$

for all $(x, x_1, \ldots, x_{n-1}) \in M \times L_1 \times \ldots \times L_{n-1}$, then there exists a linear n-functional F on $E \times L_1 \times \ldots \times L_{n-1}$ such that

$$|F(x, x_1, \ldots, x_{n-1})| \leq p(x, x_1, \ldots, x_{n-1})$$

for all $(x, x_1, \ldots, x_{n-1}) \in E \times L_1 \times \ldots \times L_{n-1}$, and

$$F(x, x_1, \ldots, x_{n-1}) = f(x, x_1, \ldots, x_{n-1})$$

for all $(x, x_1, \ldots, x_{n-1}) \in M \times L_1 \times \ldots \times L_{n-1}$.

Proof. Let $\dim L_1 = \dim L_2 = \cdots = \dim L_{n-1} = 1$. In this case the theorem is true in view of Theorem 2.1.

Now, let $\dim L_1 = k_1$, $\dim L_2 = \cdots = \dim L_{n-1} = 1$, $k_1 > 1$, and

$$L_1 = [e_1^1, \ldots, e_{k_1}^1],\ \ L_2 = [e_1^2], \ldots, L_{n-1} = [e_1^{n-1}].$$

Define $f_1, f_2, \ldots, f_{k_1}$ on

$$M \times [e_1^1] \times L_2 \times \ldots \times L_{n-1},$$
$$M \times [e_2^1] \times L_2 \times \ldots \times L_{n-1}, \ldots, M \times [e_{k_1}^1] \times L_2 \times \ldots \times L_{n-1},$$

respectively by

$$f_1(x, \lambda_l^1 e_l^1, \lambda_2 e_1^2, \ldots, \lambda_{n-1} e_1^{n-1}) = f(x, \lambda_l^1 e_l^1, \lambda_2 e_1^2, \ldots, \lambda_{n-1} e_1^{n-1}),$$

where $l = 1, \ldots, k_1$ and

$$(x, \lambda_l^1 e_l^1, \lambda_2 e_1^2, \ldots, \lambda_{n-1} e_1^{n-1}) \in M \times [e_l^1] \times L_2 \times \ldots \times L_{n-1}.$$

Then each f_l, $l = 1, 2, \ldots, k_1$ is a linear n-functional on

$$M \times [e_1^1] \times L_2 \times \ldots \times L_{n-1}$$

and for each

$$(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}) \in M \times [e_l^1] \times [e_l^2] \times \ldots \times [e_l^{n-1}],$$

we have

$$|f_l(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})| \le p(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}).$$

Using Theorem 2.1, it follows that there exists $F_1, F_2, \ldots, F_{k_1}$ on

$$E \times [e_1^1] \times L_2 \times \ldots \times L_{n-1},$$
$$E \times [e_2^1] \times L_2 \times \ldots \times L_{n-1}, \ldots, E \times [e_{k_1}^1] \times L_2 \times \ldots \times L_{n-1}$$

respectively, satisfying for $l = 1, 2, \ldots, k_1$

$$|F_l(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})| \le p(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

for each

$$(x, \lambda_l^1 e_l^1, \lambda_2 e_1^2, \ldots, \lambda_{n-1} e_l^{n-1}) \in E \times [e_1^1] \times L_2 \times \ldots \times L_{n-1},$$

and

$$F_l(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}) = f(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

for each $(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_1^{n-1}) \in M \times [e_1^1] \times L_2 \times \ldots \times L_{n-1}$.

Now, define F on $E \times L_1 \times \ldots \times L_{n-1}$ by

$$F\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right) = \sum_{l=1}^{k_1} F_l(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

for

$$\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right) \in E \times L_1 \times L_2 \times \ldots \times L_{n-1}.$$

Then F is a linear n-functional on $E \times L_1 \times \ldots \times L_{n-1}$ and for

$$\left(x, \sum_{l=1}^{k_1} \lambda_l^l e_l^l, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right) \in M \times L_l \times L_2 \times \ldots \times L_{n-1}$$

we have,

$$F\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right)$$

$$= \sum_{l=1}^{k_1} F_l(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

$$= \sum_{l=1}^{k_1} f_l(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

$$= \sum_{l=1}^{k_1} f(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

$$= f\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right)$$

as f is a linear n-functional on $M \times L_1 \times \ldots \times L_{n-1}$. Again, for

$$\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right) \in E \times L_l \times L_2 \times \ldots \times L_{n-1},$$

we have

$$\left|F\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right)\right|$$

$$= \left|\sum_{l=1}^{k_1} F_1(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})\right|$$

$$\leq \sum_{l=1}^{k_1} p(x, \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1})$$

$$= p\left(x, \sum_{l=1}^{k_1} \lambda_l^1 e_l^1, \lambda_2 e_l^2, \ldots, \lambda_{n-1} e_l^{n-1}\right)$$

as p possesses property P_R satisfying (2.12) along $L_1, \ldots, L_{n-1}$. This establishes the theorem in the case $\dim L_1 = k_1 > 1$ and $\dim L_2 = \cdots = \dim L_{n-1} = 1$.

Now, let $\dim L_1 = k_1 > 1$, $\dim L_2 = k_2 > 1$ and $\dim L_3 = \dim L_4 = \cdots = \dim L_{n-1} = 1$, where $L_2 = [e_1^2, \ldots, e_{k_2}^2]$, $L_3 = [e_1^3], \ldots, L_{n-1} = [e_1^{n-1}]$. As in the pervious case, we define k_2 linear n-functionals $g_1, g_2, \ldots, g_{k_2}$, on

$M \times L_1 \times [e_l^2] \times [e_l^3] \times \ldots \times [e_l^{n-1}]$, $M \times L_1 \times [e_2^2] \times [e_l^3] \times \ldots \times [e_l^{n-1}], \ldots,$
$M \times L_1 \times [e_{k_2}^2] \times [e_l^3] \times \ldots \times [e_l^{n-1}]$ respectively, for $l = 1, 2 \ldots k_2$, by

$$g_l(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_l^{n-1}) = f(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_l^{n-1})$$

for each

$$x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_l^{n-1}) \in M \times L_1 \times [e_l^2] \times [e_l^3] \times \ldots \times [e_1^{n-1}].$$

Then, for each $l = 1, 2, \ldots, k_2$, g_l is a linear n-functional on $M \times L_1 \times [e_1^2] \times L_3 \times \ldots \times L_{n-1}$ and

$$|g_l(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_l^{n-1})| \leq p(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_l^{n-1})$$

for each

$$(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_l^{n-1}) \in M \times L_l \times [e_l^2] \times [e_l^3] \times \ldots \times [e_1^{n-1}].$$

Then from what we proved in the previous case, it follows that exists k_2 linear n-functionals G_l, $l = 1, 2, \ldots, k_2$, defined on $E \times L_1 \times [e_l^2] \times [e_l^3 \times \ldots \times [e_1^{n-1}]$, respectively, satisfying

$$|G_l(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1})| \leq p(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1})$$

for each

$$(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3 \ldots, \lambda_{n-1} e_1^{n-1}) \in E \times L_1 \times [e_l^2] \times [e_l^3] \times \ldots \times [e_l^{n-1}]$$

and

$$G_l(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3 \ldots, \lambda_{n-1} e_1^{n-1}) = f(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1})$$

for each

$$(x, y, \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}) \in M \times L_1 \times [e_l^2] \times [e_l^3] \times \ldots \times [e_l^{n-1}].$$

Now, define G on $E \times L_1 \times L_2 \times \ldots \times L_{n-1}$ by

$$G\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right)$$

$$= \sum_{l=1}^{k_2} G_l(x, y, \ \lambda_l^2 e_l,^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1})$$

for

$$\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1})\right) \in E \times L_1 \times L_2 \times \ldots \times L_{n-1}.$$

Then G is a linear n-functional on $M \times L_1 \times L_2 \times \ldots \times l_{n-1}$ and for

$$\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right) \in M \times L_1 \times L_2 \times \ldots \times L_{n-1},$$

we have

$$
\begin{aligned}
&G\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right) \\
&= \sum_{l=1}^{k_2} G_l(x, y, \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}) \\
&\sum_{l=1}^{k_2} g_l(x, y, \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}) \\
&= \sum_{l=1}^{k_2} f(x, y, \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}) \\
&= f\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right)
\end{aligned}
$$

as f is a linear n-functional on $M \times L_1 \times L_2 \times \ldots \times L_{n-1}$.

Again, for

$$
\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right) \in E \times L_1 \times L_2 \times L_3 \times \ldots \times L_{n-1}
$$

$$
\begin{aligned}
&\left|G\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right)\right| \\
&= \left|\sum_{l=1}^{k_2} G_l(x, y, \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1})\right| \\
&\leq \sum_{l=1}^{k_2} p(x, y, \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}) \\
&= p\left(x, y, \sum_{l=1}^{k_2} \lambda_l^2 e_l,{}^2, \lambda_3 e_l^3, \ldots, \lambda_{n-1} e_1^{n-1}\right)
\end{aligned}
$$

as p possesses property P_R satisfying (2.12) along $L_1, L_2, \ldots L_{n-1}$. This proves the theorem for the case $\dim L_1 = k_1 > 1$ $\dim L_2 = K_2 > 1$ and $\dim L_3 = \dim L_4 = \cdots = \dim L_{n-1} = 1$.

Proceeding in the same manner for the case $\dim L_1 = k_1 > 1$, $\dim L_2 = k_2 > 1$ $\dim L_3 = k_3 > 1$, $\dim L_4 = \dim L_5 = \cdots = \dim L_{n-1} = 1$, and so on the theorem follows for the case, $\dim L_1 = k_1$, $\dim L_2 = k_2, \ldots, \dim L_{n-1} = k_{n-1}$.

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ПРОШИРУВАЊЕ НА n-ФУНКЦИОНАЛИ ГЕНЕРИРАНИ ОД СЕМИ n-НОРМИ

Neeraj Srivastava, S. Bhattacharya and S. N. Lal

Резиме

Во оваа работа добиено е проширување на теоремата за комплексно генерирани n-полунорми.

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