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AN IMPROVEMENT OF CAUCHY-BUNIAKOWSKY-SCHWARZ'S INEQUALITY

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Abstract. A refinement of the well-known inequality due to Cauchy-Buniakowsky-Schwarz and some natural applications are given.

Let $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$ be n-tuples of real numbers and $p=(p_1,\dots,p_n)\in R_+^n$, i.e., $p_i \geq 0$ $(i=1,\dots,n)$. The following inequality is well-known in literature as Cauchy-Buniakowsky-Schwarz's inequality

$$\sum_{i=1}^{n} p_i a_i^2 \cdot \sum_{i=1}^{n} p_i b_i^2 \geq \left(\sum_{i=1}^{n} p_i a_i b_i\right)^2. \tag{1}$$

If $p_i > 0$ $(i=1,\dots,n)$, then the equality holds in (1) if there exists a real number r so that $a_i=rb_i$ for all $i=1,\dots,n$.

The main aim of this note is to give an improvement of (1) as follows. For other recent results in connection with (1) we refere to [1]-[5] where further references are given.

Theorem. Let $a,b\in R^n$ and $p,q\in R_+^n$ with $p \geq q$, i.e., $p_i \geq q_i$ $(i=1,\dots,n)$. Then one has the inequality:

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} - \left|\sum_{i=1}^{n} p_i a_i b_i\right| \geq \\ \geq \left(\sum_{i=1}^{n} q_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i b_i^2\right)^{1/2} - \left|\sum_{i=1}^{n} q_i a_i b_i\right| \geq 0.\end{aligned} \tag{2}$$

Proof. By Cauchy-Buniakowsky-Schwarz's inequality we can state

$$\sum_{i=1}^{n} (p_i-q_i)a_i^2 \sum_{i=1}^{n} (p_i-q_i)b_i^2 \geq \left[\sum_{i=1}^{n} (p_i-q_i)a_i b_i\right]^2. \tag{3}$$

On the other hand, a simple computation shows that:

$$\left[\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} - \left(\sum_{i=1}^{n} q_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i b_i^2\right)^{1/2}\right] \geq$$
$$\geq \left(\sum_{i=1}^{n} p_i a_i^2 - \sum_{i=1}^{n} q_i a_i^2\right)\left(\sum_{i=1}^{n} p_i b_i^2 - \sum_{i=1}^{n} q_i b_i^2\right). \tag{4}$$

Since

$$\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} \geq \left(\sum_{i=1}^{n} q_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i b_i^2\right)^{1/2}$$

and

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} q_i a_i b_i\right| \geq \left|\sum_{i=1}^{n} p_i a_i b_i\right| - \left|\sum_{i=1}^{n} q_i a_i b_i\right|$$

hence, by (3) and (4), we can write:

$$\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} - \left(\sum_{i=1}^{n} q_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i b_i^2\right)^{1/2} \geq$$
$$\geq \left|\sum_{i=1}^{n} p_i a_i b_i\right| - \left|\sum_{i=1}^{n} q_i a_i b_i\right|$$

which is exactly (2). The proof is then finished.

Remark 1. If we assume that $p > q > 0$, i.e., $p_i > q_i > 0$ for all $i=1,\ldots,n$, then the equality holds in all inequalities in (2) simultaneously if there exists an $r \in R$ so that $a_i = rb_i$ $(i=1,\ldots,n)$.

Remark 2. It is easily to see, in the above assumptions for $a,b,p,q \in R^n$, that we have:

$$\sum_{i=1}^{n} p_i a_i^2 \cdot \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 \geq \sum_{i=1}^{n} q_i a_i^2 \cdot \sum_{i=1}^{n} q_i b_i^2 - \left(\sum_{i=1}^{n} q_i a_i b_i^2\right)^2 \geq 0 \tag{5}$$

which is obvious by Lagrange's identity. Consequently, our result from (2) gives another refinement for Cauchy-Buniakowsky-Schwarz's inequality than that included in (5).

Corollary 1. Let a,b be n-tuples of real numbers and denote $S^n(1) := \{x \in R^n \mid 0 \leq x_i \leq 1,\ i=1,\ldots,n\}$.

Then we have inequality:

$$\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} - \left|\sum_{i=1}^{n} a_i b_i\right| =$$

$$= \sup_{x\in S^n(1)} \left[\left(\sum_{i=1}^{n} x_i a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} x_i b_i^2\right)^{1/2} - \left|\sum_{i=1}^{n} x_i a_i b_i\right| \right] \geq 0.$$

Corollary 2. Let a,b,p,q be as in the above theorem. Then we have the following refinement of Minkowski's result:

$$\left[\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} \right]^2 - \sum_{i=1}^{n} p_i (a_i+b_i)^2 \geq$$

$$\geq \left[\left(\sum_{i=1}^{n} q_i a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} q_i b_i^2\right)^{1/2} \right]^2 - \sum_{i=1}^{n} q_i (a_i+b_i)^2 \geq 0.$$

Applications 1. Suppose that $a_i \geq 0$ $(i=1,\ldots,n)$ and $r \geq s \geq 0$. Then we have the inequality

$$\left(\sum_{i=1}^{n} a_i^r\right)^{1/2} \left(\sum_{i=1}^{n} a_i^{r+2}\right)^{1/2} - \left(\sum_{i=1}^{n} a_i^s\right)^{1/2} \left(\sum_{i=1}^{n} a_i^{s+2}\right)^{1/2} \geq$$

$$\geq \sum_{i=1}^{n} (a_i^{r+1} - a_i^{s+1}) \geq 0.$$

Applications 2. If $a_i > 0$ $(i=1,\ldots,n)$ and $p_i \geq q_i \geq 0$, then one has the inequality:

$$\left(\sum_{i=1}^{n} p_i a_i\right)^{1/2} \left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right)^{1/2} - \left(\sum_{i=1}^{n} q_i a_i\right)^{1/2} \left(\sum_{i=1}^{n} \frac{q_i}{a_i}\right)^{1/2} \geq$$

$$\geq \sum_{i=1}^{n} (p_i - q_i) \geq 0.$$

Applications 3. For all $\alpha=(\alpha_1,\ldots,\alpha_n)\in R^n$ and $a,b\in R^n$, we have the inequality:

$$\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} - \left|\sum_{i=1}^{n} a_i b_i\right| \geq$$

$$\geq \left(\sum_{i=1}^{n} a_i^2 \mathrm{trig}^2(\alpha_i)\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2 \mathrm{trig}^2(\alpha_i)\right)^{1/2} - \left|\sum_{i=1}^{n} a_i b_i \mathrm{trig}^2(\alpha_i)\right| \geq 0$$

where $\mathrm{trig}(x) = \sin x$ or $\cos x$ for $x\in R$.

R E F E R E N C E S

[1] Dragomir, S.S.: On an Inequality of Tiberiu Popoviciu's type (Romanian), G.M. Metodica (Bucuresti), 8 (1987), 124-126

[2] Dragomir, S.S.: Inequality of Cauchy-Buniakowsky-Schwarz's type for isotonic linear functionals (Romanian), G.M. Metodica, (Bucuresti), 9 (1988), 162-166

[3] Dragomir, S.S.: On an Inequality of Tiberiu Popoviciu (5), Hin. Sem. on Funct. Eq. Approx. and Convex, Cluj, 1989, 139-146

[4] Dragomir, S.S.: On Cauchy-Buniakowsky-Schwarz inequality for isotonic functionals, Sem. Opt. Theary, Babes-Bolyai, Univ. Cluj, 8 (1989), 27-34

[5] Sever, S.S., Arslanagić, Š.Z.: A Refinement of Cauchy-Buniakowsky-Schwarz Inequality for Real Numbers, Radovi mat. Vol. 7 (1991), 2 (to appear)

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JEDNO POBOLJŠANJE NEJEDNAKOSTI KOŠI-BUNJAKOVSKI-ŠVARCA

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