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GENERALIZED METRICS - (n,m,ρ)-METRICS

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Abstract. In this paper we introduce the notion of generalized, i.e. (n,m,ρ)-metrics, which is a generalization of the usual notion for metrics, and which coincide with it for $n=2$, $m=1$ and ρ the identity relation. In the case $m=1$, we use the notation (n,ρ)-metric. With this notions, the area of triangles in the plane is a $(3,\rho)$-metric, and the volume of tetrahedra in the space is a $(4,\rho)$-metric.

The notion of partitons of type n, introduced by J. Hartmanis, have been connected with the notion of generalized equivalence relation by H.E. Pickett in [1]. Several generalization of equivalence relation have been given in [2], [3] and [4]. A generalization metric, i.e. a $(n+1)$-metric in <Nm,E> nets has been introduced by J. Ušan in [5]. In our joint paper with A. Mandak we have examined a generalized metric, its induced (n,m)-balls and topologies on incidence structures. All of this led me to the introduction of the notions of (n,m)-equivalences and (n,m,ρ)-metrics, given in this paper. These generalized metrics induce certain topologies on unions of symmetric products of M. More about the properties of these generalized metrics and induced topologies will appear in a subsequent paper. At the end of this short introduction, I would like to thank the referee, for the helpful remarks about the results in this area.

For the rest of the paper, let n,m be two positive integers, such that $n-m=k\geq 1$, and let M be a nonempty set.

Let M^n denote the n^{th} Cartesian power of M. We will use the notation $x=a_1a_2\ldots a_n$ or just $x=a_1^n$ instead of $x=(a_1,a_2,\ldots,a_n)$ for elements $x\in M^n$. For $x\in M$, we denote the element $(x,x,\ldots,x)\in M^n$ by x^n.

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Definition 1. The n-fold permutation product of M, i.e. the n^{th}-symmetric power of M, is the set $M^{(n)} = M^n/\approx$, where $\approx$ is the equivalence relation defined on M^n by

$$x_1^n \approx y_1^n \iff x_1,\ldots,x_n \text{ is permutation of } y_1,\ldots,y_n. \tag{1}$$

We will use the same notation $x=a_1^n$ for elements in $M^{(n)}$ keeping in mind that $a_1^n=b_1^n$ in $M^{(n)}$, for $a_i,b_i \in M$, if and only if (iff) $b_1,b_2,\ldots,b_n$ is a permutation of $a_1,a_2,\ldots,a_n$. Let $\pi_n: Q^n \to Q^{(n)}$ be the natural projection. Note that $\pi_n(a_1^n)=\pi_n(b_1^n)$ iff $b_1,b_2,\ldots,b_n$ is a permutation of $a_1,a_2,\ldots,a_n$, i.e. $a_1^n = b_1^n$ in $Q^{(n)}$.

Definition 2. A subset ρ of $M^{(n)}$ is called symmetric n-relation on M. A symmetric n-relation on M is called reflexive n-relation on M if for each $a \in M$, $a^n=a\ldots a$ is in ρ. A symmetric n-relation on M is called transitive (n,m)-relation on M, i.e. (n,m)-transitive, if for each $x \in M^{(n)}$, $b \in M^{(m)}$,

$$(ub \in \rho \text{ for each } u \in M^{(k)} \text{ with } uv=x) \text{ implies } x \in \rho. \tag{2}$$

A reflexive n-relation on M which is (n,m)-transitive is called (n,m)-equivalence on M. Instead of saying transitive (n,1)-relation and (n,1)-equivalence, we say only transitive n-relation on M, and n-equivalence on M.

Example 1. (1) The set $\Delta=\{x^n \mid x \in M\}$ is an (n,m)-equivalence on M for each $1 \le m < n$.

(2) The set Col={(A,B,C) | A,B,C are colinear points in E^2} is a (3,t)-equivalence on E^2, for t=1,2, where E^2 is the euclidean plane.

(3) The set Com={(A,B,C,D) | A,B,C,D are complanar in E^3} is a (4,t)-equivalence on E^3, for t=1,2,3, where E^3 is the euclidean 3-dimensional space.

(4) Let M be a finite dimensional, real or complex, vector space. Then the set Liz=$\{x_1^n \mid x_1-x_2, x_1-x_3,\ldots,x_1-x_n$ are linearly dependent vectors$\}$ is an (n-m)-equivalence on M for every $1 \le m < n$. ◊

Definition 3. A map $d: M^{(n)} \to R_0^+$, which satisfies the following axioms:

(i) $d(x) = 0$ iff $x \in \rho$; and

(ii) For each $a \in M^{(m)}$, $d(x) \leq \sum_{x=uv} d(ua)$;

where R_0^+ is the set of non-negative real numbers, and ρ is an (n,m)-equivalence on M, is said to be an (n,m,ρ)-*metric on* M, and the pair (M,d) is said to be (n,m,ρ)-*metric space*. The sum in (ii) is over all $u \in M^{(n-m)}$ such that there is a $v \in M^{(m)}$ with x=uv, i.e. the sum is over all parts $u \in M^{(n-m)}$ of x. In the case m=1, instead of saying (n,1,ρ)-metric we say only (n,ρ)-metric. When there is no ambiguity about the (n,m)-equivalence ρ, we omit it and write only (n,m)-metric instead of (n,m,ρ)-metric and n-metric instead (n,ρ)-metric.

With the above notions, the notion of a (2,Δ)-metric is the same with the usual notion of metric.

Example 2. Let Δ be the (n,m)-equivalence defined in Example 1, (1), and let $d: M^{(n)} \to R_0^+$ be defined by $d(x)=0$ iff $x \in \Delta$. Then it is easy to check that d is an (n,m,Δ)-metric, and so, (M,d) is an (n,m,Δ)-metric space. We call this (n,m,Δ)-metric and (n,m,Δ)-metric space, *discrete* (n,m)-*metric* and *discrete* (n,m)-*metric space*. ◊

Example 3. Let $P: (E^2)^{(3)} \to R_0^+$ and $V: (E^3)^{(4)} \to R_0^+$ be defined by:

P(A,B,C)=the area of the triangle determined by the three points A,B and C; and

V(A,B,C,D)=the volume of the tetrahedron determined by the four points A,B,C and D.

In the case when A,B,C are colinear, P(A,B,C)=0, and when A,B,C,D are complanar, V(A,B,C,D)=0.

Then, it can be checked that P is a (3,Col)-metric on E^2 and V is a (4,Com)-metric on E^3, i.e. (E^2,P) is a (3,Col)-metric space, and (E^3,V) is a (4,Com)-metric space. ◊

R E F E R E N C E S

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ОБОПШТЕНИ МЕТРИКИ - (n,m,ρ)-МЕТРИКИ

Дончо Димовски

Р е з и м е

Во оваа работа е воведен поимот за обопштени, т.е. (n,m,ρ)-метрики, кој е обопштување на поимот за обична метрика, и кој се совпаѓа со него за n=2, m=1 и ρ идентичната релација. Во случајот m=1, го употребуваме поимот (n,ρ)-метрика. Со овој поим, плоштина на триаголници во рамнина е (3,ρ)-метрика, а волумен на тетраедри во простор е (4,ρ)-метрика.