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ON A THREE-DIMENSIONAL NONLINEAR DIFFERENTIAL SYSTEM

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Abstract

Insights on a three dimensional nonlinear system are presented in this paper. The system possesses three equilibrium points one of which is the origin $O(0,0,0)$ whose analysis in the degenerate case is performed. It is also pointed out that $O(0,0,0)$ undergoes pitchfork bifurcations.

1. Introduction

In the last time were studied more polynomial differential systems with polynomials degree 2 or 3, defined in $\mathbf{R}^3$, dependent sensitive to the initial conditions. Of such systems we recall here, the Lorenz system [1],

$$\dot{x} = a(y-x), \quad \dot{y} = cx - y - xz, \quad \dot{z} = -bz + xy, \tag{1}$$

the Chen system [6]

$$\dot{x} = a(y-x), \quad \dot{y} = (c-a)x + cy - xz, \quad \dot{z} = -bz + xy, \tag{2}$$

the Lü system [2],

$$\dot{x} = a(y-x), \quad \dot{y} = cy - xz, \quad \dot{z} = -bz + xy, \tag{3}$$

the Rössler system [8],

$$\dot{x} = -x - z, \quad \dot{y} = x + ay, \quad \dot{z} = c - bz + xz, \tag{4}$$

the Bloch system,

$$\dot{x} = -bx + y, \quad \dot{y} = -x - by + bcyz, \quad \dot{z} = ab(1 - z) - bcy^2, \tag{5}$$

and the Agiza system

$$\dot{x} = -ax + by + yz, \quad \dot{y} = bx - ay + xz, \quad \dot{z} = 1 - xy, \tag{6}$$

where a, b, c are real parameters.

The system considered in this paper is given by:

$$\dot{x} = a(y - x), \quad \dot{y} = (c - a)x - axz, \quad \dot{z} = -bz + xy, \tag{7}$$

with a, b, c real parameters and $a \neq 0$. Call it the T system. Some results regarding the T system are already presented in [3] and [4]. Compared with the Lü system introduced in [2], the system T allows a larger possibility in choosing the parameters of the system and, consequently, it displays a more complex dynamics.

2. Analysis of the equilibria

Because the dynamics of the system is characterized by the existence and the number of the equilibrium points, their type of stability, we study in the following the equilibrium points of the system T.

Proposition 2.1. *If $\frac{b}{a}(c - a) > 0$, then the system T possesses three equilibrium isolated points:*
$O(0,0,0), E_1(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}), E_2(-\sqrt{\frac{b}{a}(c-a)}, -\sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a})$, *and for $b \neq 0$, $\frac{b}{a}(c - a) \leq 0$ the system T has only one isolated equilibrium point, $O(0,0,0)$.*

Theorem 2.1. *For $b \neq 0$ the following statements are true:*
a) *If $(a > 0, b > 0, c \leq a)$, then $O(0,0,0)$ is asymptotically stable,*
b) *If $(b<0)$ or $(a < 0)$ or $(a>0, c > a)$, then $O(0,0,0)$ is unstable.*

Proof. We present the proof only for the degenerate case, the other cases being easier.

The linear matrix associated to the system T in $O(0,0,0)$ is: $J_0 = \begin{pmatrix} -a & a & 0 \\ c-a & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}$ with the characteristic polynomial:

$$f(\lambda) = -(\lambda + b)\left(\lambda^2 + a\lambda + a^2 - ac\right).$$

We observe that if $a = c$ the eigenvalues are $-b, -c, 0$, so the equilibrium point $O(0,0,0)$ is not anymore a hyperbolic point but an equilibrium degenerate point. To study the dynamics in this case, we have to transform the system T such that to be able to apply the center manifold theory. Denote $\beta := a - a_0$, with $a_0 := c$.

The corresponding eigenvectors to the eigenvalues $-b$, $-c$, 0 are $(0,0,1)$, $(1,0,0)$ and $(1,1,0)$.

Using the linear transformation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

the system T reads:

$$\begin{cases} \dot{u} = -bu + vw + w^2 \\ \dot{v} = -cv + \beta v + \beta w + cuv + cuw \\ \dot{w} = -\beta v - \beta w - cuv - cuw \end{cases} \tag{8}$$

or, in normal form:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -b & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{9}$$

with

$f_1 = vw + w^2$
$f_2 = \beta v + \beta w + cuv + cuw$
$f_3 = -\beta v - \beta w - cuv - cuw.$

Suppose that, the system T depends on the parameter β and consider this parameter as a new variable of the system. Therefore, from (9) one gets:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} -b & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \beta \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \end{pmatrix}. \tag{10}$$

By the existence theorem of the center manifold theory, there exists a center manifold for (10) in a neighborhood of the origin, which can be expressed locally as follows:

$$W^c_{loc}(O) = \{(u,v,w,\beta) \in \mathbf{R}^4 \,|\, u = f(w,\beta), v = g(w,\beta), |w| \ll 1, |\beta| \ll 1\} \tag{11}$$

with $f(0,0) = g(0,0) = 0, f'_w(0,0) = f'_\beta(0,0) = g'_w(0,0) = g'_\beta(0,0) = 0.$

Determine now the center manifold $W^c_{loc}(O)$.
Using the Taylor expansions for f and g in $(0,0)$ we have:

$$\begin{aligned} u &= f(w,\beta) = a_1 w^2 + a_2 w\beta + a_3 \beta^2 + \cdots \\ v &= g(w,\beta) = b_1 w^2 + b_2 w\beta + b_3 \beta^2 + \cdots \end{aligned} \tag{12}$$

Because $W^c_{loc}(O)$ is invariant under the dynamics of (10), one gets:

$$\begin{aligned} \dot{u} &= f'_w \dot{w} = 2a_1 w\dot{w} + a_2 \dot{w}\beta + \cdots \\ \dot{v} &= g'_w \dot{w} = 2b_1 w\dot{w} + b_2 \dot{w}\beta + \cdots \end{aligned} \tag{13}$$

and

$$\dot{w} = -\beta v - \beta w - cuv - cuw = -\beta w + \cdots \tag{14}$$

Replacing (14) in (13) we have:

$$\begin{aligned} \dot{u} &= -2\beta a_1 w^2 - a_2 w\beta^2 + \cdots \\ \dot{v} &= -2\beta b_1 w^2 - b_2 w\beta^2 + \cdots \end{aligned} \tag{15}$$

On the other hand, from (8) and (12) we have that:

$$\begin{aligned} \dot{u} &= -ba_1w^2 + w^2 - ba_2w\beta - ba_3\beta^2 + b_1w^3 + b_2w^2\beta + wb_3\beta^2 + \cdots \\ \dot{v} &= -cb_1w^2 + \beta w - cb_2w\beta - cb_3\beta^2 + \beta b_1 w^2 + b_2 w\beta^2 + \cdots \end{aligned} \tag{16}$$

Equating the terms of w^2, $w\beta$ and β^2 in (15) and (16) we find the coefficients: $a_1 = \frac{1}{b}$, $a_2 = a_3 = 0$ and $b_2 = \frac{1}{c}$, $b_1 = b_3 = 0$.
Consequently:

$$\begin{aligned} f(w,\beta) &= \frac{1}{b}w^2 + \cdots \\ g(w,\beta) &= \frac{1}{c}w\beta + \cdots \end{aligned} \tag{17}$$

Hence, the vector field reduced to the central manifold is:

$$\dot{w} = -\beta w - \frac{c}{b}w^3 - \frac{1}{c}w\beta^2 - \frac{1}{b}w^3\beta + \cdots, \quad \dot{\beta} = 0. \tag{18}$$

Now we observe that, if $\beta = 0$, equivalently to $a = c$, the point $w = 0$ is asymptotically stable in (8), so $(u, v, w) = (0, 0, 0)$ is asymptotically stable in (8), and $(x, y, z) = (0, 0, 0)$ is asymptotically stable in (7).

Let us investigate now the stability of the equilibria

$$E_{1,2}(\pm\sqrt{\frac{b}{a}(c-a)}, \pm\sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}),$$

for $\frac{b}{a}(c-a) > 0$.

Because the system T is invariant under the linear transformation $(x, y, z) \to (-x, -y, z)$, it is sufficient to study the equilibrium point E_1.

Using the transformation $(x, y, z) \to (X_1, Y_1, Z_1)$,

$$x = X_1 + \sqrt{\frac{b}{a}(c-a)}, \; y = Y_1 + \sqrt{\frac{b}{a}(c-a)}, \; z = Z_1 + \frac{c-a}{a} \tag{19}$$

the system T leads to:

$$\begin{aligned} &\dot{X}_1 = a(Y_1 - X_1), \; \dot{Y}_1 = -a\sqrt{\frac{b}{a}(c-a)}Z_1 - aX_1Z_1, \\ &\dot{Z}_1 = \sqrt{\frac{b}{a}(c-a)}(X_1 + Y_1) - bZ_1 + X_1Y_1 \end{aligned} \tag{20}$$

so we study the system (20) in $O(0,0,0)$. Denoting by $x_0 = \sqrt{\frac{b}{a}(c-a)}$, the linear matrix of the system (20) at the origin is:

$$J = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & -ax_0 \\ x_0 & x_0 & -b \end{pmatrix},$$

having the equation associated to the characteristic polynomial given by

$$\lambda^3 + \lambda^2(a+b) + bc\lambda + 2ab(c-a) = 0. \tag{21}$$

From Routh-Hurwitz conditions, this equations has all roots with negative real parts if:

$A > 0, C > 0$ and $AB - C > 0$ where $A = a+b, B = bc, C = 2ab(c-a)$, equivalently to:

$$a+b > 0,\ ab(c-a) > 0,\ b(2a^2 + bc - ac) > 0 \tag{22}$$

Therefore, we have the following result:

Theorem 2.2. *If the above conditions* (22) *are fulfilled, the equilibrium point* $E_1(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a})$ *is asymptotically stable.*

3. Pitchfork bifurcations of the system T

Consider the parameter a as bifurcation parameter.

a) *Bifurcations in* $O(0,0,0)$

In the last section we studied the stability of the equilibrium $O(0,0,0)$ and we showed, using the central manifold theory, that in the case $a = a_0 = c$, $O(0,0,0)$ is asymptotically stable and the vector field restricted to the central manifold is:

$$w' = -\beta w - \frac{1}{c} w\beta^2 - \frac{c}{b} w^3 - \frac{1}{b} w^3 \beta + \cdots$$

with $\beta = a - a_0$.

From the bifurcations theory, we observe now that the conditions such that the equilibrium point $(w,\beta) = (0,0)$ to undergo a pitchfork bifurcation at $\beta = 0$ are fulfilled.

Indeed, denoting $G(w,\beta) = -\beta w - \frac{1}{c} w\beta^2 - \frac{c}{b} w^3$ observe that

$G(0,0) = 0, \frac{\partial G}{\partial w}\ |_{(0,0)} = 0, \frac{\partial G}{\partial \beta}\ |_{(0,0)} = 0, \frac{\partial^2 G}{\partial w^2}\ |_{(0,0)} = 0, \frac{\partial^2 G}{\partial w \partial \beta}\ |_{(0,0)} = -1 \neq 0, \frac{\partial^3 G}{\partial w^3}\ |_{(0,0)} = -\frac{c}{b} \neq 0.$

Consequently, we have the following result:

Proposition 3.1. *If* $\beta = a - a_0 = 0$ *the equilibrium* $O(0,0,0)$ *of the system* T *undergoes a pitchfork bifurcation, that generate the asymptotic stable equilibrium point* $O(0,0,0)$ *if* $a > a_0$, *and for* $a < a_0$ *three equilibria:* $O(0,0,0)$ *(unstable),* $E_{1,2}(\pm\sqrt{\frac{b}{a}(c-a)}, \pm\sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a})$ *(locally stable).*

Remark that, the equilibrium $O(0,0,0)$ can not undergo a Hopf bifurcation because the roots of the characteristic polynomial of the Jacobi matrix of the system T in $O(0,0,0)$ are $\lambda_1 = -b$, $\lambda_2 = \frac{1}{2}\left(-a - \sqrt{4ac - 3a^2}\right)$, $\lambda_3 = \frac{1}{2}\left(-a + \sqrt{4ac - 3a^2}\right)$ and the last two roots can not be purely imaginary because $a \neq 0$.
b) *Bifurcations of the equilibria E_1 and E_2.*

We observe that, the characteristic polynomial of the system T in E_1 is given:

$$f(\lambda) = \lambda^3 + \lambda^2(a+b) + bc\lambda + 2ab(c-a) = 0 \tag{23}$$

Because $ab(c-a) > 0$, the system T does not display pitchfork bifurcations but Hopf bifurcations in these points.

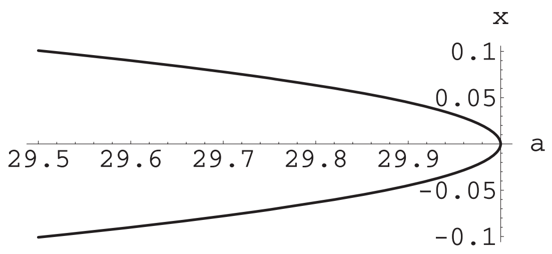

Figure 1: The $a - x$ diagram of pitchfork bifurcation of the system T for $b = 0.6, c = 30$.

Figure 2: a) Orbit of the Lü system for $a = 36, b = 3, c = 19$ and the initial condition $(-1, 0.1, 4)$ (left); b) Orbit of the T system for $(a, b, c) = (2.1, 0.6, 30)$ and the initial condition $(0.1, -0.3, 0.2)$ (right).

4. Conclusions

In this paper we further investigated a nonlinear three-dimensional differential system. The system possesses three equilibrium points, the origin $O(0,0,0)$ and another two points. The stability of $O(0,0,0)$ in the degenerate case is analyzed *via* the central manifold theory. In the origin, the system displays a pitchfork bifurcation and in the other two equilibrium points a Hopf bifurcation [5].

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ЗА ТРИДИМЕНЗИОНАЛЕН НЕЛИНЕАРЕН ДИФЕРЕНЦИЈАЛЕН СИСТЕМ

Gheorghe Tigan

Р е з и м е

Во овој труд се претставени внатрешни својства на тридеимензионален нелинеарен систем. Системот содржи три точки на рамнотежа од кои една е координатниот почеток $O(0,0,0)$ за која се изведени анализи во дегенерираниот случај. Исто така е посочено дека $O(0,0,0)$ се потчинува на "pitchfork" бифуркации.

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