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SOME BLOCK DESIGNS CONSTRUCTED FROM RESOLVABLE DESIGNS

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Abstract. Let $\mathcal{D}$ be a resolvable $2-(v,k,\lambda)$ design, and $\mathcal{D}'$ be a $2-(v',k',\lambda')$ design, such that $v' = \frac{v}{k}$. Further, let r and r' be replication numbers of a point in $\mathcal{D}$ and $\mathcal{D}'$, respectively. Shrikhande and Raghavarao proved that then there exists a $2-(v'',k'',\lambda'')$ design $\mathcal{D}''$, such that $v'' = v$, $k'' = kk'$ and $\lambda'' = r'\lambda + (r-\lambda)\lambda'$. If $\mathcal{D}'$ is resolvable, then $\mathcal{D}''$ is also resolvable. Applying this result, we construct block designs from some series of designs. Further, we discuss a construction of resolvable 3-designs.

1. Introduction

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A t-(v,k,λ) design is a finite incidence structure $(\mathcal{P},\mathcal{B},I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1.: $|\mathcal{P}| = v$;
2.: every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3.: every t elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Elements of the set $\mathcal{P}$ are called points and elements of the set $\mathcal{B}$ are called blocks. 2-designs are often called block designs. If $|\mathcal{P}| = |\mathcal{B}|$ then the block design is called symmetric. A symmetric $2-(v,k,1)$ design is called a projective plane. In a 2-(v,k,λ) design every point is incident with exactly $r = \dfrac{\lambda(v-1)}{k-1}$ blocks (see [1, Theorem 2.10 p. 10]), and r is called the replication number of a design. The number $r-\lambda$ is called the order of a $2-(v,k,\lambda)$ design. If a block design is symmetric, then $r = k$.

A parallel class or resolution class in a design is a set of blocks that partition the point set. A resolvable block design is a block design whose blocks can be partitioned into parallel classes. Resolvable block designs are frequently used in design of experiments, especially when the entire experiment can not be completed at one time or when there is a risk that the experiment may be prematurely terminated.

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In this paper we present a construction of block designs from resolvable designs, based on a result of Shrikhande and Raghavarao [7]. In some cases constructed designs are also resolvable. Further, we discuss a construction of resolvable 3-designs.

2. Construction of 2-designs

We prove the following theorem, based on a construction by Shrikhande and Raghavarao (see [7]):

Theorem 1. *Let $\mathcal{D}$ be a resolvable $2-(v,k,\lambda)$ design, and $\mathcal{D}'$ be a $2-(v',k',\lambda')$ design, such that $v'=\frac{v}{k}$. Further, let r and r' be replication numbers of a point in $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then there exists a $2-(v'',k'',\lambda'')$ design $\mathcal{D}''$, such that $v''=v$, $k''=kk'$ and $\lambda''=r'\lambda+(r-\lambda)\lambda'$. If $\mathcal{D}'$ is resolvable, then $\mathcal{D}''$ is also resolvable.*

Proof. Let $\mathcal{D}$ be a resolvable block design $(\mathcal{P},\mathcal{B},I)$, and

$$\mathcal{B}=\{x_{i,j} \mid 1\le i\le r,\ 1\le j\le v'\},$$

where $x_{i,j}$ is the j^{th} block in the i^{th} parallel class. Further, let $\mathcal{D}'$ be a block design $(\mathcal{P}',\mathcal{B}',I')$ and $\mathcal{B}'=\{x'_1,\dots,x'_{b'}\}$. Blocks of $\mathcal{D}'$, given as sets of points, are $x'_j=\{P_{j_1},\dots,P_{j_{k'}}\}$, $j=1,\dots,b'$. Let the set of points of $\mathcal{D}''$ be $\mathcal{P}$, the same as for the design $\mathcal{D}$. From each parallel class of the design $\mathcal{D}$ we construct b' blocks of $\mathcal{D}''$. From i^{th} parallel class we obtain the following blocks of $\mathcal{D}''$:

$$x''_{i,j}=x_{i,j_1}\cup x_{i,j_2}\cup\ldots\cup x_{i,j_{k'}}, \qquad j=1,\dots,b'.$$

Clearly, the number of points of $\mathcal{D}''$ is v, and each block from $\mathcal{D}''$ consists of kk' points. Two points P_1 and P_2 from $\mathcal{P}$ are incident with λ block of the design $\mathcal{D}$, and each block of $\mathcal{D}$ is a subset of r' blocks of $\mathcal{D}''$. That gives us $r'\lambda$ blocks of $\mathcal{D}''$ that contain P_1 and P_2. Further, the point P_1 lies on $r-\lambda$ blocks of $\mathcal{D}$ that do not contain P_2, hence in $r-\lambda$ parallel classes of $\mathcal{D}$ points P_1 and P_2 belong to different blocks. Two blocks form a parallel class of $\mathcal{D}$ lie in λ' blocks of $\mathcal{D}''$. That gives us further $(r-\lambda)\lambda'$ blocks of $\mathcal{D}''$ that contain P_1 and P_2. Therefore, there are $r'\lambda+(r-\lambda)\lambda'$ blocks from $\mathcal{D}''$ incident with both P_1 and P_2.

If $\mathcal{D}'$ is a resolvable design, then the definition of blocks of $\mathcal{D}''$ shows that each parallel class of $\mathcal{D}'$ induces a parallel class of $\mathcal{D}''$. □

Remark 2.1: If b and b' are numbers of blocks in the designs $\mathcal{D}$ and $\mathcal{D}'$, respectively, then $r''=rr'$ is replication number of the design $\mathcal{D}''$ and $b''=b\frac{r'}{k'}=\frac{bb'}{v'}=rb'$ is the number of blocks in $\mathcal{D}''$.

Note that the design $\mathcal{D}''$ can have repeated blocks.

Theorem 2. *Let $\mathcal{D}$ be a resolvable $2-(v,k,1)$ design, and $\mathcal{D}'$ be a $2-(v',k',\lambda')$ design, such that $v'=\frac{v}{k}$. Further, let $\mathcal{D}''$ be a $2-(v'',k'',\lambda'')$ design constructed as described in Theorem 1. If $k'<k$, then $\mathcal{D}''$ is a block design without repeated blocks.*

Proof. Two blocks from $\mathcal{D}$ intersect in at most one point, hence two blocks from the design $\mathcal{D}''$ intersect in at most $(k')^2$ points. □

Remark 2.2: Let $\mathcal{D}$ be a resolvable $2-(v,k,\lambda)$ design with replication number r, and $\mathcal{D}'_1,\ldots,\mathcal{D}'_r$ be $2-(v',k',\lambda')$ designs with replication number r', and $v'=\frac{v}{k}$. Let the blocks of $\mathcal{D}'_i$ be $x'^i_j=\{P_{i_{j_1}},\ldots,P_{i_{j_{k'}}}\}$, $j=1,\ldots,b'$. Similarly as in the proof of Theorem 1, we can construct a block design $\mathcal{D}''$ with parameters $(v,kk',r'\lambda+(r-\lambda)\lambda')$ in such a way that from i^{th} parallel class of $\mathcal{D}$ we construct the following blocks of $\mathcal{D}''$:

$$x''_{i,j}=x_{i,i_{j_1}}\cup x_{i,i_{j_2}}\cup\ldots\cup x_{i,i_{j_{k'}}},\qquad j=1,\ldots,b'.$$

In that way, taking different block designs $\mathcal{D}'_1,\ldots,\mathcal{D}'_r$, one can construct mutually non-isomorphic $2-(v,kk',r'\lambda+(r-\lambda)\lambda')$ designs. Even interchanging a design $\mathcal{D}'_k$ from the set $\{\mathcal{D}'_1,\ldots,\mathcal{D}'_r\}$ with a design $\overline{\mathcal{D}}'_k$ isomorphic to $\mathcal{D}'_k$, we can produce non-isomorphic $2-(v,kk',r'\lambda+(r-\lambda)\lambda')$ designs.

In sections 3, 4 and 5 we apply Theorem 1 to construct block designs from some well-known series of designs.

3. Some 2-designs with $k=9$

It is well-known that a $(v,3,1)$ design, so called Steiner triple system, exists if and only if $v\equiv 1,3\ (mod\ 6)$. Ray-Chaudhuri and Wilson proved the existence for resolvable $(v,3,1)$ designs for every $v\equiv 3\ (mod\ 6)$ (see [6]). Further, Hanani, Ray-Chaudhuri and Wilson in [3] proved the existence for resolvable $(v,4,1)$ designs for every $v\equiv 4\ (mod\ 12)$.

Corollary 2.1. *Let $v=18t+3$, where t is a positive integer. Then there exists a $2-(v,9,12t)$ design.*

Proof. Since $v\equiv 3\ (mod\ 6)$ and $v'=\frac{v}{3}=6t+1\equiv 1\ (mod\ 6)$, there exist a $2-(v',3,1)$ design, and a resolvable $2-(v,3,1)$ design. Replication number of a resolvable $2-(v,3,1)$ design is $r=9t+1$, and replication number of a $2-(v',3,1)$ design is $r'=3t$. Theorem 1 implies that there exists a $2-(v,9,\lambda'')$ design, where $\lambda''=3t+9t=12t$. □

Corollary 2.2. *Let $v=18t+9$, where t is a positive integer. Then there exists a resolvable $2-(v,9,12t+7)$ design.*

Proof. Since $v\equiv 3\ (mod\ 6)$ and $v'=\frac{v}{3}=6t+3\equiv 3\ (mod\ 6)$, there exist resolvable designs with parameters $2-(v,3,1)$ and $2-(v',3,1)$. Replication number of a resolvable $2-(v,3,1)$ design is $r=9t+7$, and replication number of a resolvable $2-(v',3,1)$ design is $r'=3t+1$. Theorem 1 leads us to the conclusion that there exists a $2-(v,9,\lambda'')$ design, where $\lambda''=3t+1+9t+6=12t+7$. □

4. Some examples related to 1-factorization

Let S be a set, $r=2n$ an even number and $|S|=r$. Then there exist $r-1$ partitions of S into 2-subsets, such that every pair of partitions is disjoint. In other words, there exists a 1-factorization of K_{2n} for all n. That means that for all positive integer n there is a resolvable $2-(2n,2,1)$ design. Blocks of this resolvable design are all 2-subsets of the set of points, and replication number is $2n-1$.

Corollary 2.3. *Let $v = 24t - 8$, where t is a positive integer. Then there exists a resolvable $2-(v, 8, 14t-7)$ design.*

Proof. Obviously, $v \equiv 4$ $(mod\ 12)$, hence there exists a resolvable $2-(v, 4, 1)$ design with replication number $r = 8t-3$. Since $v' = \frac{v}{4} = 6t-2$ is an even number, there is a resolvable $2-(v', 2, 1)$ design with replication number $r' = v' - 1$. Therefore, there exists a resolvable $2-(v, 8, 14t-7)$ design. □

Corollary 2.4. *Let $v = 24t + 8$, where t is a positive integer. Then there exists a resolvable $2-(v, 8, 28t+7)$ design.*

Proof. Since v is even, there is a resolvable $2-(v, 2, 1)$ design with replication number $r = v-1$. The number of blocks in a parallel class is $v' = \frac{v}{2} = 12t+4$, and therefore there exists a resolvable 2-design with parameters $(v', 4, 1)$ and replication number $r' = 4t+1$. That impies the existence of a resolvable $2-(v, 8, 28t+7)$ design. □

Corollary 2.5. *Let $v = 12t + 6$, where t is a positive integer. Then there exists a resolvable $2-(v, 6, 15t+5)$ design.*

Proof. Since v is even, there is a resolvable $2-(v, 2, 1)$ design with replication number $r = v - 1 = 12t + 5$. The number of blocks in one parallel class is $v' = \frac{v}{2} = 6t + 3$, and therefore there exists a resolvable Steiner triple system on v' points, *i.e.* a resolvable 2-design with parameters $(6t+3, 3, 1)$. Replication number of that resolvable Steiner triple system is $r' = 3t+1$. Therefore, there exists a resolvable 2-design with parameters $(v, 6, 15t+5)$ □

Corollary 2.6. *Let $v = 12t + 2$, where t is a positive integer. Then there exists a $2-(v, 6, 15t)$ design.*

Proof. The number v is even, hence there is a resolvable $2-(v, 2, 1)$ design with replication number $r = v - 1 = 12t + 1$. The number of blocks in each parallel class is $v' = \frac{v}{2} = 6t+1$, so there exists a Steiner triple system $(6t+1, 3, 1)$, with replication number $r' = 3t$. It follows from Theorem 1 that there exists a 2-design with parameters $(v, 6, 15t)$ □

5. Examples related to projective planes

It is well known that a symmetric $(n^2+n+1, n+1, 1)$ design, *e.g.* a projective plane of order n, exists for every prime power n.

Corollary 2.7. *Let n be a prime power. Then there exists a block design with parameters $(3n^2+3n+3, 3n+3, 3\frac{n^2+n}{2}+n+1)$.*

Proof. Since n^2+n is even, $3n^2+3n+3 \equiv 3$ $(mod\ 6)$. Therefore, there exists a resolvable $2-(3n^2+3n+3, 3, 1)$ design. Replication number of that block design is $r = 3\frac{n^2+n}{2}+1$. Applying Theorem 1 to a resolvable $2-(3n^2+3n+3, 3, 1)$ design and a projective plane of order n, *i.e.* a block design with parameters

$(n^2+n+1, n+1, 1)$, we get that there exists a block design with parameters $(3n^2+3n+3, 3n+3, 3\frac{n^2+n}{2}+n+1)$. □

Corollary 2.8. *For every positive integer t there exists a block design with parameters* $(4\cdot 3^{2t}+4\cdot 3^t+4, 4\cdot 3^t+4, 4\cdot 3^{2t-1}+3^t+4\cdot 3^{t-1}+1)$.

Proof. Because $4\cdot 3^{2t}+4\cdot 3^t+4 \equiv 4\ (mod\ 12)$ there is a resolvable $2-(4\cdot 3^{2t}+4\cdot 3^t+4, 4, 1)$ design, with replication number $r = 4\cdot 3^{2t-1}+4\cdot 3^{t-1}+1$. This fact and the existence of a symmetric block design with parameters $(3^{2t}+3^t+1, 3^t+1, 1)$ proves the statement. □

Corollary 2.9. *Let n be a prime power. Then there exists a block design with parameters* $(2n^2+2n+2, 2n+2, 2n^2+3n+1)$.

Proof. $2n^2+2n+2$ is an even number, so there exists a resolvable $2-(2n^2+2n+2, 2, 1)$ design with replication number $r = 2n^2+2n+1$. Further, there exists a symmetric $(n^2+n+1, n+1, 1)$ design, hence there exists a $2-(2n^2+2n+2, 2n+2, 2n^2+3n+1)$ design. □

Remark 5.1: Naturally, one can apply Theorem 1 to some other block designs, for example to the following series of designs:

- A resolvable $(v, 3, 2)$ design exists if and only if $v \equiv 0, 1\ (mod\ 6)$ and $v \neq 6$ (see [2]).
- An affine plane of order n is a block design of the form $(n^2, n, 1)$. An affine plane of order n exists if and only if a projective plane of order n exists.
- There exist quasi-residual nonresolvable 2-$(3^n7, 3^{n-1}7, (3^{n-1}7-1)/2)$ designs, for $n \geq 2$ (see [8]).

6. Construction of 3-designs

Let $\mathcal{D}$ be a $t-(v, k, \lambda_t)$ design and let $s < t$ be a positive integer. Then $\mathcal{D}$ is also an $s-(v, k, \lambda_s)$ design, where

$$\lambda_s = \lambda_t \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Theorem 3. *Let $\mathcal{D}$ be a resolvable $3-(v, k, \lambda_3)$ design, and $\mathcal{D}'$ be a $3-(v', k', \lambda_3')$ design, such that $v' = \frac{v}{k}$. Further, let r and r' be replication numbers of a point in $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then there exists a $3-(v'', k'', \lambda_3'')$ design $\mathcal{D}''$, such that $v'' = v$, $k'' = kk'$ and $\lambda_3'' = \lambda_3 r' + 3(\lambda_2-\lambda_3)\lambda_2' + (r-\lambda_3-3(\lambda_2-\lambda_3))\lambda_3'$. If $\mathcal{D}'$ is resolvable, then $\mathcal{D}''$ is also resolvable.*

Proof. Let us construct a design $\mathcal{D}''$ as described in Teorem 1, and let P_1, P_2 and P_3 be pairwise distinct points of the design $\mathcal{D}$. We have to show that there are $\lambda_3 r' + 3(\lambda_2-\lambda_3)\lambda_2' + (r-\lambda_3-3(\lambda_2-\lambda_3))\lambda_3'$ blocks of $\mathcal{D}''$ that are incident with P_1, P_2 and P_3.

Points P_1, P_2 and P_3 are incident with λ_3 block of the design $\mathcal{D}$, and each block of $\mathcal{D}$ is a subset of r' blocks of $\mathcal{D}''$. That gives us $\lambda_3 r'$ blocks of $\mathcal{D}''$ that contain P_1, P_2 and P_3.

There are $(\lambda_2 - \lambda_3)$ blocks of $\mathcal{D}$ containing two points from the set $\{P_1, P_2, P_3\}$, and not containg the third point from that set. Since there are λ'_2 blocks of $\mathcal{D}''$ that contain each pair of blocks of $\mathcal{D}$, that gives us further $3(\lambda_2 - \lambda_3)\lambda'_2$ blocks of $\mathcal{D}''$ that are incident with P_1, P_2 and P_3.

Finally, there are $(r - \lambda_3 - 3(\lambda_2 - \lambda_3))$ blocks of $\mathcal{D}$ that are incident with exactly one point from the set $\{P_1, P_2, P_3\}$, and three blocks of $\mathcal{D}$ are contained in λ'_3 blocks of $\mathcal{D}''$. That gives us additional $(r - \lambda_3 - 3(\lambda_2 - \lambda_3))\lambda'_3$ blocks of $\mathcal{D}''$ that are incident with P_1, P_2 and P_3, and completes the proof. □

It is shown in [4] and [5] that a resolvable $3-(v,4,1)$ design exists if and only if $v \equiv 4, 8 \ (mod\ 12)$.

Corollary 3.1. *Let $v = 48t + 16$, where t is a positive integer. Then there exists a resolvable $3-(v, 16, 408t^2 + 242t + 35)$ design.*

Proof. Obviously, $v \equiv 4 \ (mod\ 12)$, hence there exists a resolvable $3-(v,4,1)$ design with replication number $r = 384t^2 + 232t + 35$ and every pair of points of that design is incident with $\lambda_2 = 24t + 7$. Since $v' = \frac{v}{4} = 12t + 4$, there is a resolvable $3-(v', 4, 1)$ design with replication number $r' = 24t^2 + 10t + 1$ and $\lambda'_2 = 6t + 1$. Therefore, there exists a resolvable $3-(v, 16, 408t^2 + 242t + 35)$ design. □

Corollary 3.2. *Let $v = 48t - 16$, where t is a positive integer. Then there exists a resolvable $3-(v, 16, 408t^2 - 302t + 55)$ design.*

Proof. Since $v \equiv 8 \ (mod\ 12)$, there exists a resolvable $3-(v,4,1)$ design with replication number $r = 384t^2 - 280t + 51$ and $\lambda_2 = 24t - 9$. Further, $v' = \frac{v}{4} = 12t - 4$ so there is a resolvable $3-(v', 4, 1)$ design with replication number $r' = 24t^2 - 22t + 5$ and $\lambda'_2 = 6t - 3$. Therefore, there exists a resolvable $3-(v, 16, \lambda''_3)$ design, where $\lambda''_3 = 408t^2 - 302t + 55$. □

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