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TWO EXAMPLES OF ANALYTIC REPRESENTATION OF DISTRIBUTIONS

N. Rechkoski* and Vasko Rechkoski**

Abstract

In thus work we found which the distributions which analytic representation are the functions $\Gamma(z)$ and $\zeta(z)$.

The objective of this work is to give examples for the determining of the distribution T with a given complex function.

As it has commonly been accepted, by $D(R) = D$, we mark the space of test functions in R. D' is the dual space, i.e. the Schwartz distributions.

It is known ([1], p.76), that for every distribution $T \in D'$ there is a complex function $f(z)$ which is analytic in the plane C, except to the support "supp T" from the distribution T, and in that, it is true that:

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} [f(x+i\varepsilon)-f(x-i\varepsilon)]\varphi(x)dx = T(\varphi), \quad \varphi \in D. \tag{1}$$

The function $f(z)$ is called an analytic representation for the distribution T, and it is solely determined up to an entire function. This means that, if $f(z)$ is an analytic representation, then $f(z) + g(z)$, where $g(z)$ is an entire function, also satisfies the condition (1). Every entire function is an analytic representation for the zero distribution $0 \in D'$.

According to this, the analytic representation $f(z)$ for a given distribution T is analytic to C except to the closed set supp T from R. However, it does not function the other way round. There are functions which are analytic to C, except to a certain set from R, but they are not analytic representation to a certain distribution $T \in D'$. One such example is the function $e^{-\frac{1}{z^2}}$, $\operatorname{Im} z \neq 0$ ([1] p. 104).

In this work we give examples for the determining of a distribution when there is a given analytic function.

1. To determine the distribution whose analytic representation is the gama function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \operatorname{Re} z > 0\,. \tag{2}$$

It is known that the function (2) is analytic for $\operatorname{Re} z > 0$.

On the other hand, although (2) defines $\Gamma(z)$ only in the right half plane, the function can be extended to the whole plane with the exception of isolated poles. This extension can be carried out in several ways.

One way of making the extension is with the known relation

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for} \quad \operatorname{Re} z > 0.$$

In this work, the following method of extension catches our attention. We put

$$\Gamma(z) = \Gamma_1(z) + \Gamma_2(z),$$

where

$$\Gamma_1(z) = \int_0^1 e^{-t} t^{z-1}\, dt, \quad \Gamma_2(z) = \int_1^\infty e^{-t} t^{z-1}\, dt, \quad \operatorname{Re} z > 0.$$

Since $|t^{z-1}| = t^{\operatorname{Re} z - 1}$, $\Gamma_2(z)$ represents an entire function. Thus, to extend Γ, we need only to extend $\Gamma_1(z)$. But for $\operatorname{Re} z > 0$

$$\Gamma_1(z) = \int_0^1 \Big(1 - t + \frac{t^2}{2!} - \cdots\Big) t^{z-1}\, dt = \frac{1}{z} - \frac{1}{z+1} + \frac{1}{2!}\,\frac{1}{z+2} - \cdots$$

The above series define an analytic extension of Γ_1 to the whole plane except for the isolated poles at $0, -1, -2, \dots$. Note that

$$\operatorname{Re} z(\Gamma; -k) = \operatorname{Re} z(\Gamma_1 \colon -k) = \frac{(-1)^k}{k!}\,.$$

Now, we determine the distribution T whose analytic representation is the function Γ. Because $\Gamma_2(z)$ represents an entire function, it is a representation for the zero distribution. Thus, it would be sufficient to have a look just at

$$\Gamma_1(z) = \frac{1}{z} - \frac{1}{z+1} + \frac{1}{2!}\,\frac{1}{z+2} - \cdots$$

$$\Gamma_1(x + i\varepsilon) - \Gamma_1(x - i\varepsilon) = \Big(\frac{1}{x + i\varepsilon} - \frac{1}{x - i\varepsilon}\Big) -$$

$$-\left(\frac{1}{x+i\varepsilon+1}-\frac{1}{x-i\varepsilon+1}\right)+\frac{1}{2!}\left(\frac{1}{x+i\varepsilon+2}-\frac{1}{x-i\varepsilon+2}\right)-\cdots$$

$$-\frac{2i\varepsilon}{x^2+\varepsilon^2}+\frac{2i\varepsilon}{(x+1)^2+\varepsilon^2}+-\frac{1}{2!}\frac{2i\varepsilon}{2!(x+2)^2+\varepsilon^2}+\cdots$$

$$\frac{1}{2i}\int_{-\infty}^{\infty}[\Gamma_1(x+i\varepsilon)-\Gamma_1(x-i\varepsilon)]\varphi(x)dx=$$

$$=-\varepsilon\int_{-\infty}^{\infty}\frac{\varphi(x)}{x^2+\varepsilon^2}dx+\varepsilon\int_{-\infty}^{\infty}\frac{\varphi(x)}{(x+1)^2+\varepsilon^2}dx-\frac{\varepsilon}{2!}\int_{-\infty}^{\infty}\frac{\varphi(x)}{(x+2)^2+\varepsilon^2}+\cdots$$

A given function $\varphi\in D$ has a compact support, that is why we can put

$$\frac{1}{2i}\int_{-M}^{M}[\Gamma_1(x+i\varepsilon)-\Gamma_1(x-i\varepsilon)]\varphi(x)dx=$$

$$=-\varepsilon\int_{-M}^{M}\frac{\varphi(x)}{x^2+\varepsilon^2}dx+\frac{\varepsilon}{1!}\int_{-M}^{M}\frac{\varphi(x)}{(x+1)^2+\varepsilon^2}-\frac{\varepsilon}{2!}\int_{-M}^{M}\frac{\varphi(x)}{(x+2)^2+\varepsilon^2}+\cdots,$$

$M>0$.

Further on, we have

$$\frac{1}{2i}\int_{-M}^{M}[\Gamma_1(x+i\varepsilon)-\Gamma_1(x-i\varepsilon)]\varphi(x)dx=-\varepsilon\int_{-M}^{M}\frac{\varphi(0)+\varphi'(0)x+h_0(x)x^2}{x^2+\varepsilon^2}dx+$$

$$+\frac{\varepsilon}{1!}\int_{-M}^{M}\frac{\varphi(-1)+\varphi'(-1)(x+1)+h_1(x)(x+1)^2}{(x+1)^2+\varepsilon^2}$$

$h_0,h_1,\ldots,$ are continous functions according to the Taylor's formula.

$$-\varepsilon\int_{-M}^{M}\frac{\varphi(0)}{x^2+\varepsilon^2}dx=-\varphi(0)\operatorname{arctg}\frac{x}{\varepsilon}\Big|_{-M}^{M}=-\varphi(0)\Big(\operatorname{arctg}\frac{M}{\varepsilon}-\operatorname{arctg}\frac{-M}{\varepsilon}\Big)\quad\text{(i)}$$

$$-\varepsilon\int_{-M}^{M}\frac{\varphi'(0)x}{x^2+\varepsilon^2}dx=-\varepsilon\varphi'(0)\frac{1}{2}\log(x^2+\varepsilon^2)\Big|_{-M}^{M}\quad\text{(ii)}$$

$$-\varepsilon\int_{-M}^{M}\frac{h_0(x)x^2}{x^2+\varepsilon^2}dx\,.\quad\text{(iii)}$$

When $\varepsilon \to 0$ (i) $\to -\varphi(0)\pi$
(ii) is 0

(iii) $\left|-\varepsilon \int\limits_{-M}^{M} \frac{h_0(x)x^2}{x^2+\varepsilon^2}dx\right| \leq - - \varepsilon \int\limits_{-M}^{M} |h_0(x)|dx \to 0.$

By analogy, we get

$$\frac{\varepsilon}{1!}\int\limits_{-M}^{M} \frac{\varphi(-1)}{(x+1)^2+\varepsilon^2}dx = \frac{\varphi(-1)}{1!}\operatorname{arctg}\frac{x+1}{\varepsilon}\Big|_{-M}^{M}$$

$$\frac{\varepsilon}{1!}\int\limits_{-M}^{M} \frac{\varphi'(-1)(x+1)}{(x+1)^2+\varepsilon^2}dx = \frac{\varepsilon}{1!}\frac{\varphi'(-1)}{2}\log((x+1)^2+\varepsilon^2)\Big|_{-M}^{M}, \quad \text{etc.}$$

We get

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} [\Gamma_1(x+i\varepsilon) - \Gamma_2(x-i\varepsilon)]\varphi(x)dx =$$

$$= -\varphi(0) + \frac{\varphi(-1)}{1!} - \frac{\varphi(-2)}{2!} + \cdots = -\delta_0(\varphi) + \frac{\delta_{-1}(\varphi)}{1!}\frac{\delta_{-2}(\varphi)}{2!} + \cdots$$

From here it follows that the required distribution T is

$$T = 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\delta_{-n}}{n!}, \quad \delta_{-n}(\varphi) = \varphi(-1).$$

2. To determine the distribution whose analytic representation is the Riemann $\zeta(z)$ function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \operatorname{Re} z > 1. \tag{3}$$

The function (3) is analytic forRe $z > 1$, and it is known that it can be analytically extended.

In this work, the following method of extension catches our attention:

$$\int\limits_0^{\infty} e^{nt}t^{z-1}\,dt = \frac{1}{n^z}\int\limits_0^{\infty} e^{-t}t^{z-1}\,dt = \frac{\Gamma(z)}{n^z}.$$

By summing from n = 1 and further on, we get

$$\Gamma(z)\sum_{n=1}^{\infty}\frac{1}{n^z} = \int\limits_0^{\infty} t^{z-1}\Big(\sum_{n=1}^{\infty} e^{nt}\Big)\,dt = \int\limits_0^{\infty}\frac{t^{z-1}}{e^t-1}\,dt$$

or

$$\zeta(z) = \frac{1}{\Gamma(z)}\Big(\int_0^1 \frac{t^{z-1}}{e^t-1}\,dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt\Big). \tag{4}$$

Since $\frac{1}{\Gamma(z)}$ and $g(z) = \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt$ are both entire functions, it would be sufficient for us to take a look at the function $\int_0^1 \frac{t^{z-1}}{e^t-1}\,dt$.

The Laurent expansion for $\frac{1}{e^t-1}\,dt$ around 0 is

$$\frac{1}{e^t-1} = \frac{1}{t} + a_0 + a_1 t + \cdots$$

By integration of $\frac{t^{z-1}}{e^t-1}$ we get

$$\int_0^1 \frac{t^{z-1}}{e^t-1}\,dt = \frac{1}{z-1} + \frac{a_0}{z} + \frac{a_1}{z+1} + \cdots \tag{5}$$

The relation (5) enables analytic extension except in the poles: $1, 0, -1, \ldots$ According to (4)

$$\zeta(z) = \frac{1}{\Gamma(z)}\Big[\frac{1}{z-1} + \frac{a_0}{z} + \frac{a_1}{z+1} + \cdots + g(z)\Big]$$

Taking into consideration the zeros of $\frac{1}{\Gamma(z)}$ in $0, -1, -2, \ldots$, we get that $\zeta(z)$ has a unique pole $z = 1$ with $\operatorname{Re} z(\zeta(z), 1) = 1$.

Thus

$$\zeta(z) = \frac{1}{\Gamma(z)}\,\frac{1}{z-1} + h(z) + g(z), \quad h(z) \text{ and } g(z)$$

are entire functions. From the fact $\frac{1}{\Gamma(1)} = 1$ we get

$$\zeta(x+i\varepsilon) - \zeta(x-i\varepsilon) \sim \frac{1}{x+i\varepsilon-1} - \frac{1}{x-i\varepsilon+1} = \frac{2i\varepsilon}{(x-1)^2+\varepsilon^2}$$

and from there that $\zeta(z)$ is an analytic representation for the distribution

$$T = -\pi i \delta_1, \quad \delta_1(\varphi) = \varphi(1),$$

References

[1] G. Bremerman: *Raspredelenija kompleksnie peremenie i preobrazovanija Furje,* Izdatelstvo "Mir", Moskva, 1968.

[2] J. Bak, D. Newman: *Complex Analysis,* Springer - Verlag, Berlin 1982.

[3] L. Jantseher: *Distributionen,* Walter de Gruyter, Berlin. New York 1971.

ДВА ПРИМЕРИ НА АНАЛИТИЧКА РЕПРЕЗЕНТАЦИЈА НА ДИСТРИБУЦИИ

Н. Речкоски* и Васко Речкоски**

Резиме

Во оваа рабта определени се дистрибуциите чии што аналитични репрезентации се функциите $\Gamma(z)$ и $\zeta(z)$.

* s. Velgošti
Ohrid
R. Makedonija

** Univ. "Sv. Kliment Ohridski"
Fakultet za turizam i ugostitelstvo
Ohrid
R. Makedonija