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APROXIMATIVE METHODS FOR SOLVING SOME NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

In this paper we give approximative solutions of some differential equations whose solutions are squares or square roots of the solutions of other differential equations.

For the differential equation

$$y'' = a(x)y \tag{1}$$

where $a(x)$ is analytical function or continuous for $|x| \leq \alpha$ with initial conditions $y(0) = C_0$, $y'(0) = C_1$, the solution can be obtained ([1], [2], [3], [4]) as a limit of the sequence $\{y_k\}$, $(k \in N)$ where

$$y_{k+1} = C_0 + C_1 x + \int_0^x\int_0^x a(x) y_k(x)\, dx^2 = C_0 + C_1 x + \int_0^x\int_0^t a(v)\, y_k(v)\, dv\, dt \tag{2}$$

or

$$y_{k+1} = C_0 + C_1 x + \int_0^x (x-t)\, a(t)\, (t) y_k(t)\, dt \tag{3}$$

or

$$y_{k+1} = T\, y_k$$

where $y_0 = C_0 + C_1 x$ and respectively

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$$y_{k+1} =$$

$$=C_0\left[1+\sum_{i=1}^{k}\int_0^x\int_0^{x_1} a(x_2)\int_0^{x_2}\int_0^{x_3} a(x_4)\ldots\int_0^{x_{2i-2}}\int_0^{x_{2i-1}} a(t)dx_1\,dx_2\ldots dx_{2i-1}dt\right]+ \tag{4}$$

$$+C_1\left[x+\sum_{i=1}^{k}\int_0^x\int_0^{x_1} a(x_2)\int_0^{x_2}\int_0^{x_3} a(x_4)\ldots\int_0^{x_{2i-2}}\int_0^{x_{2i-1}} ta(t)dx_1\,dx_2\ldots dx_{2i-1}dt\right].$$

I. Let be given the nonlinear differential equation;

$$2zz'' - z'^2 - 4a(x)\,z^2 = 0 \tag{5}$$

with the initial conditions $z(0) = A > 0,\ z'(0) = B$.

The theorem holds:

Theorem **1.** The differential equation (5) has unique solution for $|x| \le \alpha$.

Proof. With the substitution $z = y^2$ (5) can be transformed in (1) with the initial conditions

$$y(0) = C_0 = \sqrt{A}, \qquad y'(0) = C_1 = \frac{B}{2\sqrt{A}}$$

$$(\text{or}\quad y(0) = C_0 = -\sqrt{A}, \qquad y'(0) = C_1 = -\frac{B}{2\sqrt{A}}\ ).$$

Because $y_{k+1} = y_0 + \int_0^x (x-t)a(t)y_k\,dt$ where $y_0 = C_0 + C_1x$, we have that

$$z_{k+1} = y_{k+1}^2 = \left(y_0 + \int_0^x (x-1)a(t)y_k(t)\,dt\right)^2 =$$

$$= y_0^2 + 2y_0\int_0^x (x-t)a(t)y_k\,dt + \left(\int_0^x (x-t)a(t)y_k\,dt\right)^2 \tag{6}$$

or

$$z_{k+1} = y_0^2 + 2y_0\int_0^x (x-t)a(t)y_k(t)\,dt +$$

$$+ 2\int_0^x\int_0^t (x-t)(t-v)a(t)y_k(t)a(v)y_k(v)\,dt\,dv +$$

$$+ 2\int_0^x\int_0^t (x-t)^2 a(t)y_k(t)a(v)y_k(v)\,dv\,dt =$$

$$= y_0^2 + 2y_0\int_0^x (x-t)a(t)\,y_k(t)\,dt +$$

$$+ 2\int_0^x\int_0^t (x-t)(x-v)a(t)\,y_k(t)\,a(v)\,y_k(v)\,dv\,dt \tag{7}$$

and y_k is given with (3) or (4). From the construction of $\{z_k\}$, it converges to solution of (5), because $\{y_k\}$ converges to solution of (1).

In that way, finding the approximate solution of (1), we find also the approximate solution of (5).

Remark **1**: The nonlinear differential equation

$$nzz'' + (1-n)z'^2 - n^2 a(x) z^2 = 0 \tag{8}$$

with the initial conditions: $z(0) = A(>0)$, $z'(0) = B$, with the substition $z = y^n$, $(n \in N)$ can be transformed in the equation (1): $y'' = a(x)y$, with $C_0 = A^{1/n}$, $C_1 = \frac{B}{n} A^{(1-n)/n}$. So the solution of the equation (8) can be obtained as a limit of the sequence $\{z_{k+1}\}$ defined with:

$$z_{k+1} = \left(y_0 + \int_0^x \int_0^t a(v) y_k \, dv \, dt\right)^n = \left(y_0 + \int_0^x (x-t) a(t) y_k \, dt\right)^n =$$
$$= y_0^n + \binom{n}{1} y_0^{n-1} \int_0^x (x-t)\, a(t) y_k \, dt + \binom{n}{2} y_0^{n-2} \left(\int_0^x (x-t) a(t) y_k \, dt\right)^2 +$$
$$+ \cdots + \binom{n}{n} \left(\int_0^x (x-t) a(t) y_k \, dt\right)^n . \tag{9}$$

The sequence $\{z_k\}$ converges to the solution of (8) because the sequence $\{y_k\}$ converges to the solution of (1).

II. Let the nonlinear differential equation be given:

$$2zz'' + 2z'^2 - a(x) z^2 = 0 \tag{10}$$

with the initial conditions $z(0) = C$, $z'(0) = D$.

With the substition $y = z^2$ we have that $y'' = a(x)y$ with the initial conditions

$$y(0) = C_0 = C^2 , \qquad y'(0) = C_1 = 2CD .$$

The Theorem holds:

Theorem **2**. For $y'' = a(x)y$ with $y(0) = C_0 \geq 0$, $y'(0) = C_1$ there exists sequence $\{y_{k+1}\}$ defined with

$$y_{k+1} = C_0 + C_1 x + \int_0^x \int_0^t a(v)\, y_k(v) \, dv \, dt \qquad (k \in N)$$

where $y_0 = C_0 + C_1 x$ for $x \in I_1 \subseteq I = \{x \| x| \leq h \leq \alpha\}$, such, that $y_{k+1} \geq 0$ and $y \geq 0$, where

$$\lim_{k \to \infty} y_{k+1} = y$$

is the unique solution of (1).

Proof: It is known ([3]) that the equation $y'' = a(x)y$ with $y(0) = C_0$, $y'(0) = C_1$, and

1) $$\underset{|x|\le\alpha}{|a(x)|} \le M$$

2) $$\underset{|x|\le\alpha}{\beta + |C_0 + C_1 x|} \le Y$$

3) $$h \le \min\left(\alpha,\ \sqrt{\frac{2\beta}{MY}}\right)$$

has unique solution y in $I = \{x \mid |x| \le h\}$, which is a limit of the sequence $\{y_{k+1}\}$, where

$$y_{k+1} = C_0 + C_1 x + \int_0^x\int_0^t a(v) y_k(v)\, dv\, dt$$

and so we have the following cases:

a) If $C_1 > 0$, $a(x) \ge 0$, then $y_0 = C_0 + C_1 \ge 0$ for $x \ge -\frac{C_0}{C_1} \ge -h$, (or $x \ge -h \ge -\frac{C_0}{C_1}$, and $Y \ge \beta + 2C_0$) $y_1 = y_0 + \int_0^x\int_0^t a(v) y_0\, dv\, dt \ge 0$, $y_1 \ge y_0$, and if $y_k \ge 0$ and $y_{k+1} - y_k \ge 0$, then $y_{k+1} = y_0 + \int_0^x\int_0^t a(v) y_k\, dv\, dt \ge 0$ and $y_{k+2} - y_{k+1} = \int_0^x\int_0^t a(v)(y_{k+1} - y_k)\, dv\, dt \ge 0$ (see Fig. 1).

b) If $C_1 < 0$, $a(x) \ge 0$, then $y_0 = C_0 + C_1 x \ge 0$ for $x \le -\frac{C_0}{C_1} \le h$, (or $x \le h < -\frac{C_0}{C_1}$ and $Y \ge \beta + 2C_0$), $y_1 = y_0 + \int_0^x\int_0^t a(v) y_0\, dt\, dv \ge 0$, $y_1 \ge y_0$ and if $y_k \ge 0$, and $y_{k+1} - y_k \ge 0$, then $y_{k+1} = y_0 + \int_0^x\int_0^t a(v) y_k\, dv\, dt \ge 0$, $y_{k+2} - y_{k+1} = \int_0^x\int_0^t a(v)(y_{k+1} - y_k)\, dv\, dt \ge 0$ (see Fig. 2).

c) If $C_1 > 0$, $a(x) = -b(x) \le 0$, respectively $b(x) \ge 0$, then $y_0 = C_0 + C_1 x \ge 0$, for $x \ge -\frac{C_0}{C_1} \ge -h$, (or $x \ge h > -\frac{C_0}{C_1}$) and

$$y_1 = y_0 + \int_0^x\int_0^{x_1} a(x_2) y_0\, dx_2\, dx_1 = y_0 - \int_0^x\int_0^{x_1} b(x_2) y_0\, dx_2\, dx_1 \le y_0$$

but

$$
\begin{aligned}
y_1 =& y_0 - \int_0^x \int_0^{x_1} b(x_2) y_0 \, dx_2 \, dx_1 \geq y_0 - M \int_0^x \int_0^{x_1} y_0 \, dx_2 \, dx_1 \geq \\
&\geq y_0 - M \left(C_0 \frac{x^2}{2!} + C_1 \frac{x^3}{3!} \right) \geq \\
&\geq C_0 \left(1 - \frac{\beta}{Y} \right) + C_1 \sqrt{\frac{2\beta}{MY}} \left(1 - \frac{\beta}{3Y} \right) = \\
=& C_0 \frac{Y - \beta}{Y} + C_1 \sqrt{\frac{2\beta}{MY}} \frac{3Y - \beta}{3Y} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
y_2 =& y_0 + \int_0^x \int_0^{x_1} a(x_2) y_1 \, dx_2 \, dx_1 = y_0 - \int_0^x \int_0^{x_1} b(x_2) y_1 \, dx_2 \, dx_1 = \\
=& y_0 - \int_0^x \int_0^{x_1} b(x_2) \left(y_0 - \int_0^{x_2} \int_0^{x_3} b(x_4) y_0 \, dx_4 \, dx_3 \right) dx_2 \, dx_1 = \\
=& y_0 - \int_0^x \int_0^{x_1} b(x_2) y_0 \, dx_2 \, dx_1 + \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_3) y_0 \, dx_4 \, dx_3 \, dx_2 \, dx_1 = \\
=& y_1 + \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_3) y_0 \, dx_4 \, dx_3 \, dx_2 \, dx_1 \geq 0 \quad \text{and} \quad y_0 \geq y_2 \geq y_1 .
\end{aligned}
$$

Let it be $y_0 \geq \cdots \geq y_{2n} \geq y_{2n+2} \geq y_{2n+1} \geq y_{2n-1} \geq \cdots \geq y_1 \geq 0$.
Then

$$
\begin{aligned}
y_{2n+3} - y_{2n+1} =& y_0 - \int_0^x \int_0^{x_1} b(x_2) y_{2n+2} \, dx_2 \, dx_1 - y_0 + \int_0^x \int_0^{x_1} b(x_2) y_{2n} \, dx_2 \, dx_1 = \\
&= \int_0^x \int_0^{x_1} b(x_2)(y_{2n} - y_{2n+2}) \, dx_2 \, dx_1 \geq 0 ,
\end{aligned}
$$

respectively $y_{2n+3} \geq y_{2n+1}$,

$$
\begin{aligned}
y_{2n+4} - y_{2n+2} =& y_0 - \int_0^x \int_0^{x_1} b(x_2) y_{2n+3} \, dx_2 \, dx_1 - y_0 + \int_0^x \int_0^{x_1} b(x_2) y_{2n+1} \, dx_2 \, dx_1 = \\
&= \int_0^x \int_0^{x_1} b(x_2)(y_{2n+1} - y_{2n+3}) \, dx_2 \, dx_1 \leq 0 ,
\end{aligned}
$$

respectively $y_{2n+4} \leq y_{2n+2}$,

$$
\begin{aligned}
y_{2n+4} - y_{2n+3} =& y_0 - \int_0^x \int_0^{x_1} b(x_2) y_{2n+3} \, dx_2 \, dx_1 - y_0 + \int_0^x \int_0^{x_1} b(x_2) y_{2n+2} \, dx_2 \, dx_1 = \\
=& \int_0^x \int_0^{x_1} b(x_2)(y_{2n+2} - y_{2n+3}) \, dx_2 \, dx_1 = \\
=& \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_4)(y_{2n} - y_{2n+1}) dx_4 \, dx_3 \, dx_2 \, dx_1 \geq 0 \quad \text{(see Fig. 3).}
\end{aligned}
$$

d) If $C_1 < 0$, $a(x) = -b(x) \le 0$ i.e. $b(x) \ge 0$,

$$y_0 = C_0 + C_1 x \ge 0\,, \quad \text{for} \quad x \le -\frac{C_0}{C_1} \le h\,, \ (\text{or } x \le h \le -\frac{C_0}{C_1})$$

$$y_1 = y_0 + \int_0^x \int_0^{x_1} a(x_2) y_0 \, dx_2 \, dx_2 = y_0 - \int_0^x \int_0^{x_1} b(x_2) y_0 \, dx_2 \, dx_1 \le y_0$$

$$y_1 = y_0 + \int_0^x \int_0^{x_1} a(x_2) y_0 \, dx_2 \, dx_1 =$$

$$= y_0 - \int_0^x \int_0^{x_1} b(x_2) y_0 \, dx_2 \, dx_1 \ge y_0 - M\left(C_0 \frac{x^2}{2!} + C_1 \frac{x^3}{3!}\right) \ge$$

$$\ge C_0 \left(1 - \frac{\beta}{Y}\right) + C_1 \sqrt{\frac{2\beta}{MY}} \left(1 - \frac{\beta}{3Y}\right) \ge 0$$

$$y_2 = y_0 + \int_0^x \int_0^{x_1} a(x_2) y_1 \, dx_2 \, dx_1 =$$

$$= y_0 - \int_0^x \int_0^{x_1} b(x_2) \left(y_0 - \int_0^{x_2} \int_0^{x_3} b(x_4) y_0 \, dx_4 \, dx_3\right) dx_2 \, dx_1 =$$

$$= y_0 - \int_0^x \int_0^{x_1} b(x_2) y_0 \, dx_2 \, dx_1 + \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_4) y_0 \, dx_4 \, dx_3 \, dx_2 \, dx_1 =$$

$$= y_1 + \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_4) y_0 \, dx_4 \, dx_3 \, dx_2 \, dx_1 \ge 0, \quad \text{and } y_0 \ge y_2 \ge y_1.$$

Let it be $y_0 \ge \cdots \ge y_{2n} \ge y_{2n+2} \ge y_{2n+1} \ge y_{2n-1} \ge \cdots \ge y_1 \ge 0$, then as in the case c), we can prove:

$$y_{2n+2} \ge y_{2n+4} \ge y_{2n+3} \ge y_{2n+1} \qquad \text{(see Fig 4).}$$

f) If $C_1 = 0$, $a(x) \ge 0$, $y_0 = C_0 > 0$,

$$y_1 = y_0 + \int_0^x \int_0^{x_1} a(x) y_0 \, dx_2 \, dx_1 = C_0 \left(1 + \int_0^x \int_0^{x_1} a(x) \, dx_2 \, dx_1\right) \ge 0\,,$$

and $y_1 \ge y_0$.

If $y_k \ge 0$, $y_k - y_{k-1} \ge 0$, then

$$y_{k+1} = y_0 + \int_0^x \int_0^{x_1} a(x) y_k \, dx_2 \, dx_1 \ge 0\,,$$

$$y_{k+1} - y_k = \int_0^x \int_0^{x_1} a(x_2)(y_k - y_{k-1}) \, dx_2 \, dx_1 \ge 0 \qquad \text{(see Fig. 5).}$$

g) If $C_1 = 0,\ a(x) = -b(x) \leq 0$, than $y_0 = C_0$,

$$y_1 = y_0 + \int_0^x \int_0^{x_1} a(x) y_0\, dx_2\, dx_1 = y_0 - \int_0^x \int_0^{x_1} b(x) y_0\, dx_2\, dx_1 =$$

$$= C_0 \left(1 - \int_0^x \int_0^{x_1} b(x_2) y_0\, dx_2\, dx_1\right) \geq C_0 \left(1 - M\frac{x^2}{2!}\right) \geq$$

$$\geq C_0 \left(1 - M\frac{2\beta}{2MY}\right) = C_0 \frac{Y - \beta}{Y} \geq 0 \qquad \text{and } y_1 \leq y_0\,,$$

$$y_2 = y_0 + \int_0^x \int_0^{x_1} a(x_2) y_1\, dx_2\, dx_1 =$$

$$= C_0 (1 - \int_0^x \int_0^{x_1} b(x_2) y_0\, dx_2\, dx_1 + \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_4) y_0\, dx_4\, dx_3\, dx_2\, dx_1) =$$

$$= C_0 (1 - \int_0^x \int_0^{x_1} b(x_2) y_0\, dx_2\, dx_1) + C_0 \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_4) y_0\, dx_4\, dx_3\, dx_2\, dx_1 =$$

$$= y_1 + C_0 \int_0^x \int_0^{x_1} b(x_2) \int_0^{x_2} \int_0^{x_3} b(x_4) y_0\, dx_4\, dx_3\, dx_2\, dx_1 \geq 0, \text{ and } y_1 \leq y_2 \leq y_0\,.$$

Let it be $y_0 \geq \cdots \geq y_{2n} \geq y_{2n+2} \geq y_{2n+1} \geq y_{2n-1} \geq \cdots \geq y_1 \geq 0$, then as in the case c), we can prove:

$$y_{2n+2} \geq y_{2n+4} \geq y_{2n+3} \geq y_{2n+1} \text{(see Fig. 6)}.$$

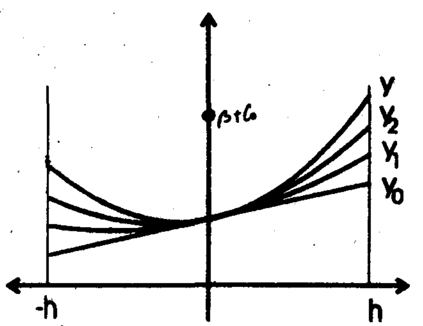

Fig. 1

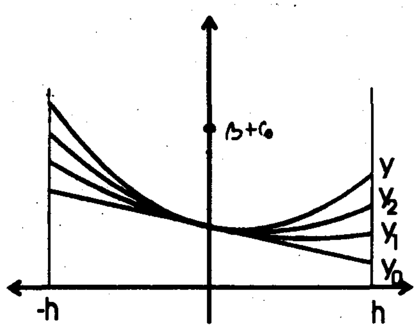

Fig. 2

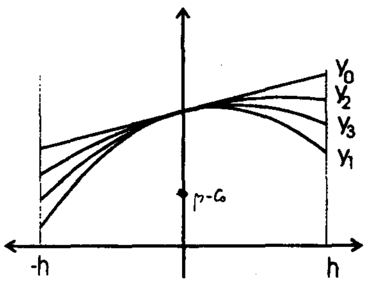

Fig. 3

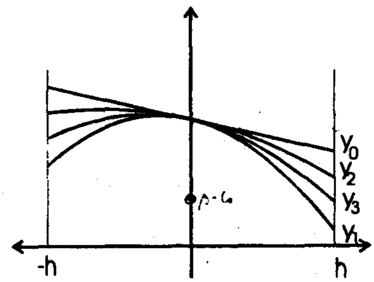

Fig. 4

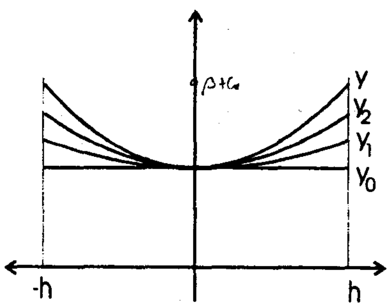

Fig. 5

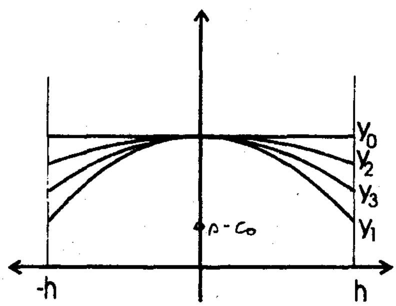

Fig. 6

Theorem **3**. The differential equation (10) has unique solution z where

$$z = \lim_{k\to\infty} z_{k+1} \qquad \text{and} \qquad z_{k+1} = \sqrt{y_{k+1}}\,.$$

The proof can be made using the construction of the sequences $\{y_k\}$ and $\{z_k\}$ where $y_{k+1} \geq 0$ and z_{k+1} is obtained in the following way:

$$\begin{aligned} z_{k+1} &= \sqrt{y_{k+1}} = \sqrt{y_0 + (y_{k+1} - y_0)} = \sqrt{y_0 + \int_0^x \int_0^t a(v) y_k \, dv \, dt} = \\ &= \sqrt{y_0} + \frac{1}{2} \int_0^x \int_0^t a(v) y_k \, dv \, dt + \qquad (11) \\ &+ \frac{1}{\sqrt{y_0}} \frac{1}{2} \left(-\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\int_0^x \int_0^t a(v) y_k \, dv \, dt\right)^2 + \cdots \end{aligned}$$

Remark **2**. If $C < 0$, there exist unique solution z^* of (10) where

$$z^* = \lim_{k\to\infty} z^*_{k+1}, \qquad \text{and} \qquad z^*_{k+1} = -\sqrt{y_{k+1}}$$

Remark **3**. Let the nonlinear differential equation be given:

$$nzz'' + n(n-1)z'^2 - a(x)z^2 = 0 \tag{12}$$

with the initial condition $z(0) = C$, $z'(0) = D$. With the substituon $y = z^n$, $(n \in N)$ the equation (12) is transformed to (1) $y'' = a(x)y$, with initial conditions

$$C_0 = C^n, \quad C_1 = nC^{n-1}D.$$

The solution of (12) can be obtained as a limit of the sequence $\{z_{k+1}\}$, where

$$z_{k+1} = \sqrt[n]{y_{k+1}} = \sqrt[n]{y_0 + (y_{k+1} - y_0)} = \sqrt[n]{y_0 + \int_0^x\int_0^t a(v)\, y_k(v)\, dv\, dt} =$$

$$= \sqrt[n]{y_0} + \frac{1}{n}\int_0^x\int_0^t a(v)\, y_k\, dv\, dt + \tag{13}$$

$$+ (1/\sqrt[n]{y_0})\,\frac{1}{2}\,\frac{1}{n}\,\frac{1-n}{n}\left(\int_0^x\int_0^t a(v)\, y_k\, dv\, dt\right)^2 + \cdots$$

and $\sqrt[n]{y_k}$ exists, because the equation (1) is selfadjoint.

Remark **4**. In this way there can be solved any nonlinear differential equation, whose solutions are squares and square roots of the solutions of any linear selfadjoint equation.

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АПРОКСИМАТИВНИ МЕТОДИ ЗА РЕШАВАЊЕ НА НЕКОИ НЕЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ

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Резиме

Во овој труд се конструирани апроксимативни методи за решавање на некои нелинеарни диференцијални равенки што се добиени со степенување и коренување на итерации. Така, (6) (односно (7), е решение на Кошиевиот проблем (5), што е квадрат од решението (2) на Кошиевиот проблем (1). Решението (9) на Кошиевиот проблем (8) е n–та степен од решението (2) на равенката (1). (11) пак е решение на (10), што е квадратен корен од решението (2) на равенката (1), обезбедено со теоремата **2** и (13) е решение на (12), што е n–ти корен од решението (2) на равенката (1).

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