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NEW SEMI-ORTHOGONALITY PROPERTY FOR THE JACOBI POLYNOMIALS AND FINITE FOURIER JACOBI EXPANSION

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Abstract. A new semi-orthogonality property for the Jacobi polynomials is given with an elementary weight function with its application.

1. Introduction

The Jacobi polynomials are orthogonal polynomials [3, p. 285, (5) and (9)] over the interval (-1,1) with respect to the weight function $(1-x)^a(1+x)^b$, if $\mathrm{Re}\,a > -1$, $\mathrm{Re}\,b > -1$.

In this paper, we present a new semi-orthogonality property for the Jacobi polynomials over the interval (-1,1) with respect to the weight function $(1-x)^a(1+x)^{b+n-m-1}$, if $\mathrm{Re}\,a > -1$, $\mathrm{Re}\,b > m-n$. As an application of our semi-orthogonality, we also present a finite Fourier Jacobi expansion.

The Jacobi polynomials are defined by the relation [2, p. 170, (16)]:

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,-n-b;a+1;\frac{x-1}{x+1}\right) \tag{1.1}$$

provided a is not a negative integer.

2. The semi-orthogonality

The semi-orthogonality to be established is

$$\int_{-1}^{1} (1-x)^a(1+x)^{b+n-m-1} P_m^{(a,b)}(x) P_n^{(a,b)}(x)\,dx \tag{2.1}$$

$$= 0, \text{ if } m < n \tag{2.1a}$$

$$= \frac{2^{a+b}\Gamma(1+a+n)\Gamma(1+b+n)}{n!\ b\Gamma(a+b+n+1)}, \text{ if } m = n \tag{2.1b}$$

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$$= \frac{2^{a+b}\Gamma(2+a+b)\Gamma(2+b+n)}{n!b(b+1)(1-b)\Gamma(a+b+n+1)}, \text{ if } m = n+1 \tag{2.1c}$$

where $Re a > -1$, $Re b > m-n$.

Proof. In view of (1.1), the integral (2.1) can be written as

$$\frac{(1+a)_m(1+a)_n}{m!2^m n!2^n}\int_{-1}^{1}(1-x)^a(1+x)^{b+2n-1}\,{}_2F_1(-m,-m-b;a+1;\tfrac{x-1}{x+1})$$

$$\times {}_2F_1(-n,-n-b;a+1;\tfrac{x-1}{x+1})\,dx =$$

$$= \frac{(1+a)_m(1+a)_n}{m!2^m n!2^n}\sum_{r=0}^{m}\frac{(-m)_r(-m-b)_r(-1)^r}{(a+1)_r r!}\sum_{u=0}^{n}\frac{(-n)_u(-n-b)_u(-1)^u}{(a+1)_u u!}$$

$$\times\int_{-1}^{1}(1-x)^{a+r+u}(1+x)^{b+2n-1-r-u}dx \tag{2.2}$$

Evaluating the last integral in (2.2) with the help of a modified form of the integral [1, p.9], viz.

$$\int_{-1}^{1}(1-x)^a(1+x)^b dx = 2^{a+b+1}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} Re a > -1,\ Re b > -1,$$

then the right hand side of (2.2) becomes

$$\frac{2^{a+b+n-m}(1+a)_m(1+a)_n}{m!n!}\sum_{r=0}^{m}\frac{(-m)_r(-m-b)_r(-1)^r\Gamma(a+r+1)\Gamma(b+2n-r)}{(a+1)_r r!\Gamma(a+b+2n+1)}$$

$$\times\ {}_3F_2\left[\begin{matrix}-n,-n-b,a+r+1;1\\ a+1,1+r-b-2n\end{matrix}\right] \tag{2.3}$$

Now applying Saalschutz's theorem [1, p. 188, (3)] to (2.3), we get

$$\frac{2^{a+b+n-m}(1+a)_m}{m!n!}\sum_{r=0}^{m}\frac{(-m)_r(-r)_n(-m-b)_r(-1)^r\,\Gamma(a+r+1)\Gamma(b+2n-r)(1+a+b+n)_n}{(a+1)_r r!\Gamma(a+b+2n+1)(n+b-r)_n} \tag{2.4}$$

If $r < n$, the numerator of (2.4) vanishes and since r runs from 0 to m, it follows that (2.4) also vanishes when $m < n$. Now, it is clear that for $m < n$ all terms of the series (2.4) vanish, which proves (2.1a).

On setting $m = n$ in (2.4), using the standard result:

$$(-r)_n = \begin{cases} \frac{(-1)^n r!}{(r-n)!}, \text{ if } 0 \le n \le r \\ 0, \text{ if } n > r \end{cases} \tag{2.5}$$

and simplifying with the help of [1, p. 3, (4)], we have

$$\int_{-1}^{1} (1-x)^a (1+x)^{b-1} \{P_n^{(a,b)}(x)\}^2 dx$$

$$= \frac{2^{a+b}\Gamma(1+a+n)\Gamma(1+b+n)}{n!\ b\Gamma(a+b+n+1)}, \text{ Rea} > -1, \text{ Reb} > 0, \tag{2.6}$$

which proves (2.1b).

In (2.4), putting m = n+1, using (2.5) and adding the resulting two terms (r=n,n+1) and simplifying with the help of [1, p. 3, (4)], we obtain

$$\int_{-1}^{1} (1-x)^a (1+x)^{b-2} P_{n+1}^{(a,b)}(x) P_n^{(a,b)}(x) dx$$

$$= \frac{2^{a+b}\Gamma(2+a+n)\Gamma(2+b+n)}{n!b(b+1)(1-b)\Gamma(a+b+n+1)}, \text{ Rea} > -1, \text{ Reb} > 1 \tag{2.7}$$

which proves (2.1c).

Note 1: On continuing as above, we can find the values of the integral (2.1) for m=n+2,n+3,n+4,... .

Note 2: In (2.6) replacing x by -x, interchanging a and b, and using the relation $P_n^{(a,b)}(x) = (-1)^n P_n^{(b,a)}(-x)$, we obtain a known result [3, p. 285, (6)].

3. Finite Fourier Jacobi expansion

Based on the relation (2.1a) and (2.1b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in series of the Jacobi polynomials. Specially if f(x) are suitable functions defined for all x, we consider for expansion of the general form

$$f(x) = \sum_{m=0}^{n} A_m (1+x)^{-m} P_m^{(a,b)}(x), \tag{3.1}$$

where the Fourier coefficients are given by

$$A_m = \frac{m!\ b\Gamma(a+b+m+1)}{2^{a+b}\Gamma(1+a+m)\Gamma(a+b+m)}\int_{-1}^{1} f(x)(1-x)^a(1+x)^{b+m-1}P_m^{(a,b)}(x)dx \qquad (3.2)$$

R E F E R E N C E S

[1] Erdélyi A., et. al.: Higher transcendental functions, Vol. 1, McGraw-Hill, New York, 1953

[2] Erdélyi A., et. al.: Higher transcendental functions, Vol. 2, McGraw-Hill, New York, 1953

[3] Erdélyi A., et. al.: Tables of integral transforms, Vol. 2, McGraw-Hill, New York, 1954

НОВО ПОЛУОРТОГОНАЛНО СВОЈСТВО ЗА JACOBI ПОЛИНОМИ И КОНЕЧНО ФУРИЕВО-ЈАКОБИ ШИРЕЊЕ

С.Д. Бајпаи

Р е з и м е

Дадено е ново полуортогонално својство за Jacobi полиноми со една елементарна тежинска функција со нејзините примени.

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