Математички Билтен
24 (L)
2000 (61–68)
Скопје, Македонија

ISSN 0351–336X

DISTRIBUTIONS GENERATED WITH BOUNDARY VALUES OF FUNCTIONS OF THE SPACE H^p, $1 \leq p < \infty$

Vesna Manova Eraković

Abstrakt.

In this work are given necessary and sufficient conditions for a regular distribution in D' to be distribution generated with a boundary function of some function from the space H^p, $1 \leq p < \infty$.

0. Introduction

0.1: Denotations which will be used in the paper

Let U denote the open unit disc in $\mathbb{C}$ i.e. $U = \{z \mid |z| < 1\}$, $T = \partial U$ and Π^+ denote the upper half plane i.e. $\Pi^+ = \{z \in \mathbb{C} \mid Imz > 0\}$.

For a given function f which is analytic on some region Ω we will write $f \in H(\Omega)$.

For a function f, $f: \Omega \to \mathbb{C}^n$, $\Omega \subseteq \mathbb{R}^n$ we define differential operator $D^\alpha = D_x^\alpha$, $x \in \mathbb{R}^n$ $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{N} \cup \{o\}$ by $D^\alpha f = D_x^\alpha f(x) = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} f(x)$, where

$$D_j = \frac{\partial}{\partial x_j} \quad j = 1, 2, \dots, n$$

So

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n .$$

$L^p(\Omega)$ is the Lebesque space of measurable functions f on Ω for which

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < \infty\,, \quad 0 \le p < \infty\,,$$

and

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \operatorname*{supess}_{x\in\Omega} |f(x)|\,.$$

$L^p_{loc}(\Omega)$ is the space of locally integrable functions on Ω, i.e. $f(x) \in L^p_{loc}(\Omega)$ if $f(x) \in L^p(\Omega')$, for every bounded subregion Ω' of Ω.

0.2. The spaces H^p defined on U and Π^+ and some of their properties

For $0 < p < \infty$, the space H^p is defined to consist of all $f(z) \in H(U)$ for which

$$\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \|f\|^p_{H^p} < \infty\,.$$

In the case $p = \infty$, $H^\infty(U)$ is the space of all bounded analytic functions $f(z)$ on U, for which the norm is defined by

$$\|f\|_{H^\infty} = \sup_{z\in U} |f(z)|\,.$$

We will give the definition of those spaces in the case of the upper half plane Π^+.

Let $f(z) \in H(\Pi^+)$ and let $0 < p < \infty$. We say that $f \in H^p(\Pi^+)$ iff

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p dx = \|f\|^p_{H^p} < \infty\,.$$

In the case $p = \infty$, $H^\infty(\Pi^+)$ is the space of all bounded analytic functions on Π^+, for which the norm is defined by

$$\|f\|_{H^\infty} = \sup_{z\in\Pi^+} |f(z)|\,.$$

It is known [5] that every function $f(z) \in H^p(\Pi^+)$, $0 < p < \infty$, for almost every $t \in \mathbf{R}$, has nontangential limit f^*, $f^*(t) \in L^p(\mathbf{R})$ satisfyng

$$\int_{-\infty}^{\infty} |f^*(t)|^p dt = \|f^*\|^p_p = \|f\|^p_{H^p}$$

and

$$\lim_{y\to 0^+} \|f(t+iy) - f^*(t)\|^p_p = 0\,.$$

0.3 Some notions of distributions

$C^{\infty}(\mathbf{R}^n)$ denote the space of all complex valued infinitely differentable functions on $\mathbf{R}^n$ and $C_0^{\infty}(\mathbf{R}^n)$ denote the subspace of $C_0^{\infty}(\mathbf{R}^n)$ that consist of those functions of $C_0^{\infty}(\mathbf{R}^n)$ which have compact support. Support of a function f, denoted by supp(f) is the closure of $\{x \mid f(x) \neq 0\}$ in $\mathbf{R}^n$. $D = D(\mathbf{R}^n)$ denote the space of $C_0^{\infty}(\mathbf{R}^n)$ functions in which convergense is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in D as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset \mathbf{R}^n$ such that supp$(\varphi_\lambda) \subseteq K$ for each λ, supp$(\varphi) \subseteq K$ and for every n-tuple α of nonnegative integers the sequence $\{D_t^{\alpha}\varphi_\lambda(t)\}$ converges to $D_t^{\alpha}\varphi(t)$ uniformly on K as $\lambda \to \lambda_0$.

$D' = D'(\mathbf{R}^n)$ is the space of all continuons linear functionals on D, where continuity means that $\varphi_\lambda \to \varphi$ in D as $\lambda \to \lambda_0$ implies $\langle T, \varphi_\lambda\rangle \to \langle T, \varphi\rangle$ as $\lambda \to \lambda_0$, $T \in D'$.

Note: $\langle T, \varphi\rangle$ denotes the value of the functional T, when it acts on the function φ.

D' is called the space of distributions.

$S = S(\mathbf{R}^n)$ will denote the space of all infinetly differentiable complex valued functions φ on $\mathbf{R}^n$ satisfying

$$\sup_{t \in \mathbf{R}^n} |t^{\beta} D^{\alpha}\varphi(t)| < \infty$$

for all n-tuples α and β of nonnegative integers. Convergence in S is defined in the folowing way: a sequence $\{\varphi_\lambda\}$ of function $\varphi_\lambda \in S$ converges to $\varphi \in S$ in S as $\lambda \to \lambda_0$ if and only if

$$\lim_{\lambda \to \lambda_0} \sup_{t \in \mathbf{R}^n} |t^{\beta} D_t^{\alpha}[\varphi_\alpha(t) - \varphi(t)]| = 0$$

for all n-tuples α and β of nonegative integers.

Again, S' is the space of all continous, linear functionals on S.

Let φ be an ellement of one of the above function spaces D or S. Let $f(x) \in L^1_{loc}(\mathbf{R}^n)$. Then the functional T_f from D (or S) to $\mathbf{C}$, defined by:

$$\langle T_f, \varphi\rangle = \int_{\mathbf{R}^n} f(t)\varphi(t)dt, \quad \varphi \in D \quad (\varphi \in S)$$

is distribution on D (or S), called regular distribution generated with f.

1. Main results

The idea for the theorem 1 and theorem 2 comes from the following theorem, that is given in [6].

Theorem. Necessary and sufficient condition for a measurable function $\varphi(e^{i\theta})$, defined on T, to coincide almost everywhere on E with boundary value $f^*(e^{i\theta})$ of some function $f(z)$ from the space H^p, $p \geq 1$, is the existance of a sequence of polynomials $\{P_n(z)\}$ such that:

1) $\{P_n(e^{i\theta})\}$ converges to $\varphi(e^{i\theta})$ almost everywhere on E.

2) $\overline{\lim_{n\to\infty}} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta < \infty$.

Theorem 1. *Let T_{f^*} be the distribution in D' generated with a boundary value $f^*(x)$ of some function $f(z)$ from the space H^p, $p \geq 1$. Then there exist sequence of polynomials $\{P_n(z)\}$, $z \in \Pi^+$ and respectivelly sequence of distributions $\{T_n\}$, $T_n \in D'$ generated with the boundary values $P_n^*(x)$ of $P_n(z)$, satisfyng*

i) $T_n \to T_{f^*}$, $n \to \infty$ *in* D'.

ii) $\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} |P_n^*(x)|^p |\varphi(x)| dx < \infty$, $\forall \varphi \in D$.

Proof. Let the conditions of T.1. be satisfied. Since $f(z) \in H^p$, it follows that there exists a constant $C > 0$, such that

$$\left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{1/p} \leq C, \quad \text{for all } x+iy \in \Pi^+ . \tag{1}$$

Since $f^*(x)$ is the boundary value of $f(z) \in H^p$ it follows that $f^*(x)$ belongs to the space $L^p(\mathbf{R}^n)$ and

$$f(x+iy) \to f^*(x) \text{ in } L^p, \text{ as } y \to 0^+, \; x+iy \in \Pi^+ . \tag{2}$$

Theorem in [3] claims that $f(x+iy) \to f^*(x)$ in S', as $y \to 0^+$, $x+iy \in \Pi^+$ i.e.

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)dx = \int_{-\infty}^{\infty} f^*(x)\varphi(x)dx, \quad x+iy \in \Pi^+, \quad \varphi \in S . \tag{3}$$

Let $\{y_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} y_n = 0$. We consider the sequence of functions $\{F_n(z)\}$, defined with $F_n(z) = f(z+iy_n)$. Then $F_n(z)$ are analytic functions on $\Pi^+ \cup \mathbf{R}$. Using the theorem of Mergelyan we get that for a compact subset K of $\Pi^+ \cup \mathbf{R}$ and for the function $F_n(z)$ there exist a polynomial $P_n(z)$, such that $|F_n(z) - P_n(z)| < \varepsilon_n$ for $z \in K$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Now we will prove (i) and (ii).

(i) Let $\varphi \in D$ and $\operatorname{supp}\varphi = K \subset \mathbf{R}$

$$|\langle T_n, \varphi\rangle - \langle T_{f^*}, \varphi\rangle| = \left| \int_{-\infty}^{\infty} P_n^*(x)\varphi(x)dx - \int_{-\infty}^{\infty} f^*(x)\varphi(x)dx \right| =$$

$$= \left| \int_{-\infty}^{\infty} [P_n^*(x) - f^*(x)]\varphi(x)dx \right| \leq$$

$$\leq \int_K |P_n^*(x) - f^*(x)| \, |\varphi(x)| dx \overset{\varphi \in D \subset S}{=}$$

$$\leq M \int_K |P_n^*(x) - f^*(x)| dx \leq M\varepsilon_n' m(K) \to 0, \quad n \to \infty,$$

where $m(K)$ is the Lebesque measure of the set K, M is positive real number and $\varepsilon'_n = \varepsilon_n + |f^*(x) - F_n(x)|$. Clearly, $\varepsilon'_n \to 0$ as $n \to \infty$.

From the above computations we conclude that $\langle T_n, \varphi\rangle \to \langle T_{f^*}, \varphi\rangle$ as $n \to \infty$, for every $\varphi \in D$.

(ii)

$$\left(\int_{-\infty}^{\infty} |P_n^*(x)|^p |\varphi(x)| dx\right)^{1/p} \le$$

$$\le \left(\int_{-\infty}^{\infty} |P_n^*(x) - F_n(x)|^p |\varphi(x)| dx\right)^{1/p} + \left(\int_{-\infty}^{\infty} |F_n(x)|^p |\varphi(x)| dx\right)^{1/p} \le$$

$$\le M^{1/p} \left(\int_K |P_n^*(x) - F_n(x)|^p dx\right)^{1/p} + \left(\int_K |F_n(x)|^p dx\right)^{1/p} \le$$

$$\le M^{1/p} \varepsilon_n m^{1/p}(K) + M^{1/p} \left(\int_K |f(x+iy_n)|^p dx\right)^{1/p} \le$$

$$\le M^{1/p} \varepsilon_n m^{1/p}(K) + M^{1/p} C \to M^{1/p} C\,, \quad n \to \infty\,.$$

It follows that

$$\int_{-\infty}^{\infty} |P_n^*(x)|^p |\varphi(x)| dx \le MC^p = C_1$$

which proves (ii).

Theorem 2. *Let φ_0 be a locally integrable function on $\mathbf{R}$ and T_{φ_0} be the distribution in D' generated with φ_0. Let there exists a sequence of polynomials $P_n(z)$, $z \in \Pi^+$ such that the following conditions are satisfied:*

i) *The sequence of distribution, generated with the boundary values $P_n^*(x)$ of $P_n(z)$ converges to T_{φ_0} in D' as $n \to \infty$.*

ii)

$$\overline{\lim_{n\to\infty}} \int_{\mathbf{R}} |P_n(x+iy)|^p \, |\varphi(x)| dx < \infty\,, \quad \forall (x+iy) \in \Pi^+\,, \quad \forall \varphi \in D\,.$$

Then there exists function $f(z) \in H(\Pi^+)$ such that

$$\int_K |f(x+iy)|^p dx < C < \infty \quad \forall (x+iy) \in \Pi^+$$

for every compact subset K of $\mathbf{R}$

and

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)dx = \langle T_{\varphi_0}, \varphi\rangle, \quad \forall \varphi \in D.$$

Proof. From (i) i.e.

$$\lim_{n\to\infty} \int_{\mathbf{R}} P_n^*(x)\varphi(x)dx = \int_{\mathbf{R}} \varphi_0(x)\varphi(x)dx, \quad \varphi \in D$$

it follows that

$$\begin{cases} \text{there exists function } f(z) \in H(\Pi^+), \text{ such that the} \\ \text{sequence of polynomials } P_n(z) \text{ converges to } f(z) \\ \text{uniformelly on compact subsets of } \Pi^+ \text{ as } n \to \infty. \end{cases} \qquad (4)$$

Note: We will give explination of (4).

Let $C_c(\mathbf{R})$ be the space of functions that vanish in infinity, $C_0(\mathbf{R})$ be the space of functions with compact support. It is known that $C_0(\mathbf{R})$ is dense in $C_c(\mathbf{R})$.

Let

$$\lim_{n\to\infty} \int_{\mathbf{R}} P_n^*(x)\varphi(x)dx = \int_{\mathbf{R}} \varphi_0(x)\varphi(x)dx \quad \varphi \in C^\infty(\mathbf{R})$$

Taking $P_y(t-x)$ (Poisson kernel) instead of $\varphi(x)$, and using integral representations for analytic functions, we get that

$$\lim_{n\to\infty} P_n(z) = f(z),$$

where

$$f(z) = \int_{\mathbf{R}} \varphi_0(x)P_y(t-x)dx.$$

Even more the convergence is uniform on compact subsets of $\mathbf{R}$.

[Similar result is given in [6]].

Now, using the fact that $C_0(\mathbf{R})$ is dense in $C_c(\mathbf{R})$, it can be proven (4).

Let $\varphi \in D$ and $\mathrm{supp}(\varphi) = K \subset \mathbf{R}$. Then

$$\lim_{y\to 0^+} \int_{\mathbf{R}} f(x+iy)\varphi(x)dx \underset{(4)}{=} \lim_{y\to 0^+} \int_K \lim_{n\to\infty} P_n(x+iy)\varphi(x)dx \underset{uc}{=}$$

$$= \lim_{n\to\infty} \lim_{y\to 0^+} \int_K P_n(x+iy)\varphi(x)dx =$$

$$= \lim_{n\to\infty} \int_{\mathbf{R}} P_n^*(x)\varphi(x)dx \underset{(i)}{=}$$

$$= \int_{\mathbf{R}} \varphi_0(x)\varphi(x)dx = \langle T_{\varphi_0}, \varphi\rangle, \quad \forall \varphi \in D.$$

So, we proved that

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)dx = \langle T_{\varphi_0}, \varphi\rangle, \quad \forall \varphi \in D.$$

Now, from (ii) we have that

$$\overline{\lim_{n\to\infty}} \int_{\mathbf{R}} |P_n(x+iy)|^p \, |\varphi(x)| dx < C < \infty, \quad \forall (x+iy) \in \Pi^+, \quad \forall \varphi \in D. \quad (5)$$

Let K be a compact subset of $\mathbf{R}$. Then there exists

$$\varphi(x) \in C_0^\infty(\mathbf{R}), \quad \varphi(x) = 1, \quad \forall x \in K.$$

Substiting $\varphi(x)$, choosen in this way, in (5), we get

$$\overline{\lim_{n\to\infty}} \int_K |P_n(x+iy))|^p dx < C < \infty. \qquad (6)$$

Now

$$\int_K |f(x+iy)|^p dx = \int_K \lim_{n\to\infty} |P_n(x+iy)|^p dx \le \overline{\lim_{n\to\infty}} \int_K |P_n(x+iy)|^p dx < C < \infty$$

i.e.

$$\int_K |f(x+iy)|^p dx \le C < \infty,$$

for every compact subset K of $\mathbf{R}$ and for every $x+iy \in \Pi^+$.

Note: Theorem 1 and Theorem 2 are valid for S', and when working in S' by using (3) it can be given another proof. But more important thing is that these theorems can be done when the functions considered in the theorems, are not from the space H^p, $p > 1$, but they belong to the Nevanlina space.

References

[1] Bremerman, G.: *Распределения, комплексные переменные и преобразования Фурье*, MIR, Moskva (1968).

[2] Carmichael, R. D. and Mitrović, D.: *Distributions and analytic funkctions*, John Wiley and Sons, Inc., New York (1988).

[3] Carmichael, R. D and Richtrs, S. P.: *Holomorfic functions in tubes which have distributional boundary value and which are H^p functions*, Siam J. Math. anal. Vol. 14, N° 3 (1983).

[4] Duren, P. L.: *Theory of H^p Spaces*, Acad. Press, New York (1970).

[5] Garnett, J.: *Bounded analytic functions*, Acad. Press, New York (1981).

[6] Privalov, I. I.: *Граничние Своиства Одозначниш Аналитическиш Функции*, Nauka, Moskva (1941).

[7] Schwartz, L.: *Theorie des distributions*, Tome I, Tome II, Herman and C^{ie}, Paris (1950, 1951).

[8] Stankovič, B. Pilipović S.: *Teorija distribucija*, PMF, Novi Sad (1983)

ДИСТРИБУЦИИ ГЕНЕРИРАНИ СО ГРАНИЧНИ ВРЕДНОСТИ НА ФУНКЦИИ ОД КЛАСАТА H^p, $1 \leq p < \infty$

Весна Манова Ераковиќ

Р е з и м е

Во оваа работа се дадени потребни и доволни услови за да регуларна дистрибуција во D' биде дистрибуција генерирана со гранична вредност на некоја функција од класата H^p, $1 \leq p < \infty$.

Faculty of Mathematics and Natural Sciences
Institute of Mathematics
St. Cyril and Methodius University
P. fah 162, 1000 Skopje,
MACEDONIA

e-mail: vesname@iunona.pmf.ukim.edu.mk