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ON THE STRUCTURE OF INITIAL OBJECTS OF FIBRE CATEGORIES

by

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Abstract

Let $\mathcal{E}$ be a cocomplete category and $P: \mathcal{E} \to \mathcal{B}$ be a colimit preserving functor having normalized split cleavage. If B is a colimit object of a diagram D in $\mathcal{B}$ over a shceme $\Sigma = (I, M, d)$, then it is shown thf the initial object of the fibre $\mathcal{E}_B$ is precisely the colimit of the diagram ID in E over Σ such that $\mathrm{ID}_{(\alpha)}$ is the initial object of $\mathcal{E}_{D(\alpha)}$ for every α in I. This result is applied to the universal complexes, free objects, tensor algebras etc.

1. **Introducton:** Let $P: \mathcal{E} \to \mathcal{B}$ be a functor. This work is the result of an investigation into the connections between the structure of an object $B \in \mathcal{B}$ and that of the initial object of the fibre $\mathcal{E}_B = p^{-1}(B)$. In this paper a reasonably natural connection has been established between the two, when $\mathcal{E}$ is cocomplete and P is a colimit preserving functor having a normalized split cleavage. Such functors are frequent enough to occur in the study of complexes over algebras, free objects, Grothendieck groups, free Lie algebras etc. We have proved that in this situation if B is the colimit of a diagram D in B over a scheme $\Sigma = (I, M, d)$, then the initial object of $\mathcal{E}_B$ is precisely the colimit of the diagram ID in $\mathcal{E}$ over Σ such that ID(α) is the initial object of $\mathcal{E}_{D(\alpha)}$, $\alpha \in I$ (provided „enough" initial objects exist). To obtain this result the adjoint functors between the category of diagrms ID and the category $\mathcal{E}_B$ are constructed.

2. **Preliminaries**: The **fibre** of a functor $P : \mathcal{E} \to \mathcal{B}$ over B is the sub-category $\mathcal{E}_B = p^{-1}(B)$ of $\mathcal{E}$ consisting of all morphisms $\varnothing$ in $\mathcal{E}$ such tha- $P(\varnothing) = i_B$, where i_B is the identity on B. Let $J_B : \mathcal{E}_B \to \mathcal{E}$ be the inclu- sion functor. A **cleavage** consits of a funtor $f^*: \mathcal{E}_B \to \mathcal{E}_{B'}$, for each morphism $f : B' \to B$ in $\mathcal{B}$ together with a natural trnsformation $\theta_f: J_{B'} f^* \to J_B$, satisfying the following axiom:

Axiom 2.1: $P(\theta_f) = f$ and *if* $\varnothing$: $E'' \to E$ satisfies $P(\varnothing) = fg$, for some g, then there is a unique $\varnothing'$: $E'' \to f^*(E)$, such that $P(\varnothing') = g$ and $\theta_{fE} \circ \varnothing' = \varnothing$.

The cleavage is **normalized** if f^* is an identity functor whenever f is an identity morphism. It is a **split cleavage** if $(fog)^* = g^* \text{ o} f^*$ and $\theta_{fogX} = \theta_{fX} \circ \theta_{gf^*(X)}$, whenever fog is defined in $\mathcal{B}$.

Throughout in this paper we assume that $P : \mathcal{E} \to \mathcal{B}$ is a colimit preserving functor, having normalized split cleavage and $\mathcal{E}$ cocomplete.

In the following dicussion D stands for a fixed diagram in $\mathcal{B}$ over a scheme $\Sigma = (I, M, d)$, having the colimit $\{f_\alpha : D(\alpha) \to B\}_{\alpha \in I}$. $\mathcal{D}$ denotes the sub-category of the category of diagrams in $\mathcal{E}$ over Σ obtained as follows:

ID $\in \mathcal{D}$ iff $P(\text{ID}(\alpha)) = D(\alpha)$ and $P(\text{ID}(m)) = D(m)$ for $\alpha \in I$ and $m \in M$. ID $\to$ ID$'$ is in $\mathcal{D}$ iff ID $(\alpha) \to$ ID$'(\alpha)$ is in $\mathcal{E}_{D(\alpha)}$ for $\alpha \in I$.

3. Colimits:

Lemma 3. 1: For $X' \in \mathcal{E}$ and an isomorphism $f : P(X') \to B$, there exist an object $X \in \mathcal{E}_B$ and na isomorphism $\psi : X' \to X$, such that $P(\psi) = f$.

Proof: Select $X = (f^{-1})^* X'$ and $\psi = \theta_{fX}$. Since the cleavage is split, normalized, $f^*(X) = X'$ and $\theta_{f^{-1}X'} = \psi^{-1}$.

Lemma 3.1 allows us to assume, that if $\{\psi_\alpha : \text{ID}(\alpha) \to X\}$ is a colimit of ID $\in \mathcal{D}$ then $X \in \mathcal{E}_B$ and $P(\psi_\alpha) = f_\alpha$. When this is done, consider a morphism ID$\to$ID$'$ in $\mathcal{D}$. Let $\{\psi_\alpha : \text{ID}(\alpha) \to X\}$ and $\{\psi_\alpha' =: \text{ID}'(\alpha) \to X'\}$ be the colimits of ID and ID$'$ respectively. Let $\psi : X \to X'$ be the unique morphism such that

$$(\text{ID}(\alpha) \to \text{ID}'(\alpha) \xrightarrow{\psi_\alpha'} X') = (\text{ID}(\alpha) \xrightarrow{\psi_\alpha} X \xrightarrow{\psi} X').$$

Since $P(\psi_\alpha) = f_\alpha = P(\psi_\alpha')$ it follows that ψ is in $\mathcal{E}_B$.

This defines the „colimit functor" F: $\mathcal{D} \to \mathcal{E}_B$ in a natural way.

On the other hand, for $X \in \mathcal{E}_B$ we construct a diagram $\overline{X}$ in $\mathcal{D}$ by setting $\overline{X}(\alpha) = f_\alpha^*(X)$ and $\overline{X}(m) = \theta_{D(m)} f_\beta^*(X)$, where $m \in M$ and $d(m) = (\alpha, \beta)$. For $\varnothing$: $X \to X'$ in $\mathcal{E}_B$, we set $\varnothing_\alpha = f_\alpha^*(\varnothing)$: $\overline{X}(\alpha) \to \overline{X'}(\alpha)$. If $m \in M$ and $d(m) = (\alpha, \beta)$ then in the following diagram (next page): Since $\theta_{f_\gamma} : J_{D(\gamma)} f_\gamma^* \to J_B$ is a natural transformation for every $\gamma \in I$, the quadrangles 1 and 2 commute. The triangles 3 and 4 commute because of the definitions of $\overline{X}(m)$ and $\overline{X'}(m)$ respectively. Hence

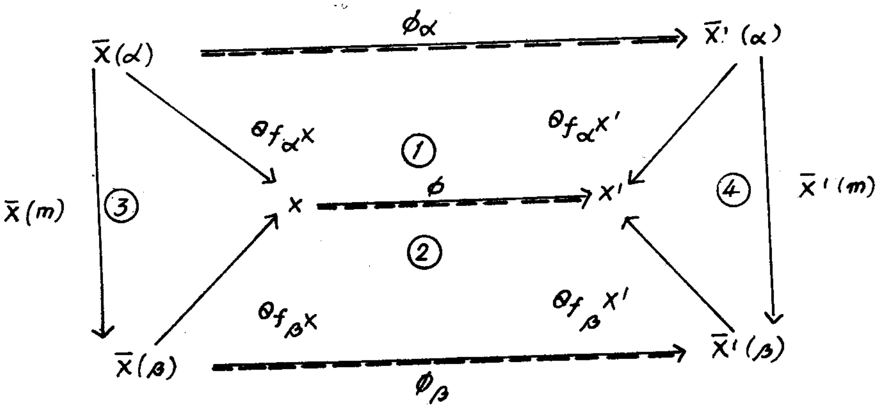
$$\theta_{f_\beta X'} \circ \varnothing_\beta \overline{X}(m) = \varnothing \circ \theta_{f_\alpha X} = \theta_{f_\beta X'} \circ \overline{X'}(m) \circ \varnothing_\alpha .$$

But since $P(\varnothing \circ \theta_{f_\alpha X}) = f_\beta \circ D(m)$, Axiom 2.1 ensures that

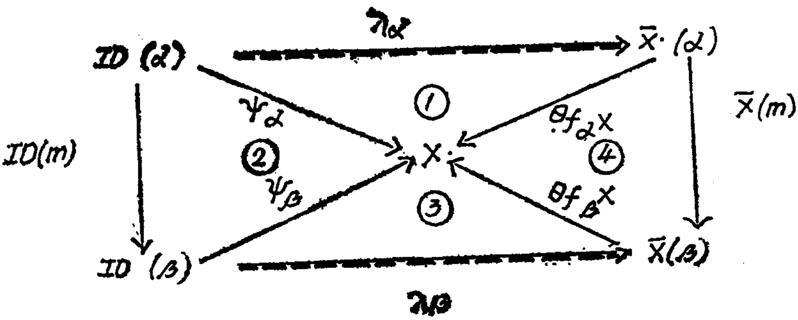
$\varnothing_\beta \circ \overline{X}(m) = \overline{X'}(m) \circ \varnothing_\alpha$. Therefore $\overline{\varnothing} = (\varnothing_\alpha)$: $\overline{X} \to \overline{X'}$ is in $\mathcal{D}$.

This leads to the functor $G\colon \mathscr{E}_B \rightarrow \mathscr{D}$ defined by $G(X)=\overline{X}$, $G(\emptyset)=\overline{\emptyset}$. Our claim is:

Theorem 3.2: $G : \mathscr{E}_B \rightarrow \mathscr{D}$ is the adjoint of $F : \mathscr{D} \rightarrow \mathscr{E}_B$.



Proof: Let $\{\psi_\alpha\colon \mathrm{ID}(\alpha) \rightarrow X\}$ be the colimit of $\mathrm{ID} \in \mathscr{D}$. $F(\mathrm{ID})=X$, $G(X)=\overline{X}$ and $\lambda_\alpha\colon \mathrm{ID}(\alpha) \rightarrow \overline{X}(\alpha)$ the unique morphism such that $\theta_{f_\alpha X} \circ \lambda_\alpha = \psi_\alpha$ offe- red by Axiom 2.1. Then for $m \in M$ and $d(m)=(\alpha, \beta)$ the triangles 1 to 4 in the following diaram commute.



Therefore $\theta_{f_\beta X} \circ \lambda_\beta \circ \mathrm{ID}(m) = \psi_\alpha = \theta_{f_\beta X} \circ \overline{X}(m) \circ \lambda_\alpha$.

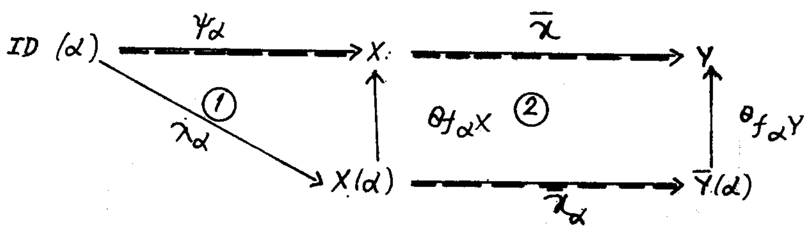
But since $P(\psi_\alpha) = f_\beta \circ D(m)$, Axiom 2.1 ensures that

$\lambda_\beta \circ \mathrm{ID}(m) = \overline{X}(m) \circ \lambda_\alpha$. Therefore $\lambda = (\lambda_\alpha)\colon \mathrm{ID} \rightarrow \overline{X}$ is in $\mathscr{D}$.

Let $Y \in \mathscr{C}_B$ and $\chi = (\chi_\alpha)$: ID $\to Y$ be any morphism in $\mathscr{D}$. Since the family $\{\mathrm{ID}(\alpha) \xrightarrow{\chi_\alpha} \overline{Y}(\alpha) \xrightarrow{\theta_{f_\alpha Y}} Y\}$ is cocompatible, there exists a unique $\overline{\chi}$: $X \to Y$ in $\mathscr{C}_B$ such that

$$\overline{\chi} \circ \psi_\alpha = \theta_{f_\alpha Y} \circ \chi_\alpha .$$

If $G(\overline{\chi}) = (\overline{\chi}_\alpha)$: $\overline{X} \to \overline{Y}$, then in the following diagram



the triangle 1 and the square 2 commute to give

$$\theta_{f_\alpha Y} \circ \overline{\chi}_\alpha \circ \lambda_\alpha = \overline{\chi} \circ \psi_\alpha = \theta_{f_\alpha Y} \circ \chi_\alpha .$$

In view of Axiom 2.1 we get

$$\chi_\alpha = \overline{\chi}_\alpha \circ \lambda_\alpha .$$

As for the uniqueness of $\overline{\chi}$ we observe that a morphism $\mu = X \to Y$ in $\mathscr{C}_B$ satisfying the equation $G(\mu) \circ \lambda = \chi$ has to satisfy the equations $\mu \circ \psi_\alpha = \theta_{f_\alpha Y} \circ \chi_\alpha$ and hence $\mu = \overline{\chi}$.

4. Initial Objects: Let B' be an arbitrary object in $\mathscr{B}$ and $u(B')$ the initial object of $\mathscr{C}_B'$. We claim:

Proposition 4.1: If $f: B' \to B$ is in $\mathscr{B}$ and $X \in \mathscr{C}_B$ then there exists a unique morphism $\varnothing$: $u(B') \to X$, satisfying $P(\varnothing) = f$.

Proof: The existence is assured by $u(B) \to f^*(X) \xrightarrow{\theta_{fX'}} X$. As for the uniqueness, if $\varnothing$: $u(B') \to X$ and ψ: $u(B') \to X$ are such that $P(\varnothing) = = f = P(\psi)$, then by Axiom 2.1, there exist $\varnothing'$: $u(B') \to f^*(X)$ and ψ': $u(B') \to f^*(X)$ such that $\theta_{fX} \circ \varnothing' = \varnothing$ and $\theta_{fX} \circ \psi' = \psi$. Since $\varnothing' = \psi'$ we have $\varnothing = \psi$.

This proposition in conjuction with Theorem 3.2 yields

Theorem 4.2: If ID $\in \mathcal{D}$ is such that ID$(\alpha) = u(D(\alpha))$ for every $\alpha \in I$, then ID is the initial object of $\mathcal{D}$ and $F(\text{ID}) = u(B)$; in other words $\varinjlim u\ (D(\alpha)) = u(\varinjlim D(\alpha))$.

If every fibre of P has an initial object (e.g. if $\mathcal{E}$ and $\mathcal{B}$ are cocomplete and P has both fibration and opfibration [1]) then in view of proposiion 4.1 every diagram D in $\mathcal{B}$ gives rise to a diagram ID in $\mathcal{E}$ such that ID$(\alpha) = u(D(\alpha))$. This leads to

Theorem 4.3: If every fibre of P has an initial object then $B = \varinjlim D(\alpha)$ implies $u(B) = \varinjlim u(D(\alpha))$.

5. **Complexes**: The categoris of Complexes over algebras offer quite an important application of Theorem 4.3. Let R be a commutative ring with unity. An R-complex is an ordered pair (X, d), of an anticommuttive graded R-algebra $X = \bigoplus_{n \geqslant 0} X_n$ and derivative $d : X \to X$ of degree 1 such that $d^2 = 0$ [2]. An R-Complex homomorphism $\varnothing: (X, d) \to (Y, \delta)$ is a graded R-algebra homomorphism $\varnothing: X \to Y$ satisfying $\varnothing \circ d = \delta \circ \varnothing$. The category $\mathcal{C}(R)$ of R-Complexes is cocomplete [3]. If $\mathcal{A}$ is the category of commutative unitary R-algebras, the projection functor P: $\mathcal{C}(R) \to \mathcal{A}$ given by

$$P((X, d)) = X_0 \text{ and } P((X, d) \to (Y, \delta)) = X_0 \to Y_0$$

clearly preserves colimits. An object of the fibre $\mathcal{C}(R)_A$ is an A-complex, a morphism an A-complex homomorphism and the initial object, the universal A-complex.

If $f : A' \to A$ is in $\mathcal{A}$ and $(X, d) \in \mathcal{C}(R)_A$ then we define,

$$f^* : \mathcal{C}(R)_A \to \mathcal{C}(R)_{A'} : (X, d) \to (X', d');$$

where $X' = A' \oplus (\bigoplus_{n \geqslant 1} X_n)$, $d' \mid A' = df$, $d' \mid \bigoplus_{n \geqslant 1} X_n = d$.

X' is an A' — algebra via the action $a' x = f(a') x$. For $\varnothing: (X, d) \to (Y, \delta)$ in $\mathcal{C}(R)_A$ we define $f^*(\varnothing) = \varnothing'$, where $\varnothing' | A' =$ identity and $\varnothing' | \bigoplus_{n \geqslant 1} X_n = \varnothing$.

Further $\theta_f: J_{A'} f^* \to J_A$ is defined by $\theta_{f(X, d)}: (X', d') \to (X, d)$ where $\theta_{f(X, d)} \mid A' = f$, $\theta_{f(X, d)} \mid \bigoplus_{n \geqslant 1} X_n =$ identity.

Straightforward computations show that the above f^* and θ_f offer a normalized split cleavage for P. Since the universal A-complex exists for every $A \in \mathcal{A}$[2], the main results of the paper [3] become a corollary to Theorem 4.3, namely if an algebra $A \in \mathcal{A}$ is a colimit of algebras $A_\alpha \in \mathcal{A}$ then the universal A-complex is the colimit of the universal A_α — complexes.

6. **Other Applications**: Let $U: \mathcal{C} \to \mathcal{K}$ be a functor which is injective on morphisms. We construct the category $\mathcal{H}$ whose objects are morphisms $\lambda: K \to U(X)$, $K \in \mathcal{K}$, $X \in \mathcal{C}$ and the morphisms $\lambda_1 \to \lambda_2$ are pairs $(f, \varnothing)$ of morphisms $f: K_1 \to K_2$ in $\mathcal{K}$ and $\varnothing: X_1 \to X_2$ in $\mathcal{C}$ satisfyng $U(\varnothing) \circ \lambda_1 = \lambda_2 \circ f$. Let $p: \mathcal{H} \to \mathcal{K}$ be the „natural projection" functor

$$P(K \to U(X)) = K \text{ and } P(f, \varnothing) = f.$$

Definition 6.1: An initial object of the fibre $\mathcal{H}_A$ is the free object over A relative to U.

The traditional free objects, the Grothendieck groups, free Lie algebras are free objects in the above sense.

If $\mathcal{C}$ and $\mathcal{K}$ are complete and if U preserves colimits then $\mathcal{H}$ is cocomplete and P preserves colimits. Moreover, P has a normalized split cleavage, defined in the obvious fashion. Therefore, it follows that colimits of free objects are freee objects.

The above construction can be extended to include tensor product, exterior product and similar objects. For exampl, if m is the category of R-modules then we construct τ to be the category of n — linear maps $\lambda: M_1 \times \ldots \times M_n \to M: M, M_1 \ldots M_n \in m$.

The morphisms $\lambda_1 \to \lambda_2$ are $n+1$ tuples $(f_1, \ldots, f_n, f)$ such that $f: M \to N$, $f_i: M_i \to N_i$ for $i = 1, \ldots n$ and

$$\lambda_2 (f_1(x_1), \ldots, f_n(x_n)) = f \circ \lambda_1 (x_1, \ldots, x_n) \text{ for } x_i \in M_i.$$

The projection functor $P: \tau \to m \times \ldots \times m$ and the cleavage are defined suitably.

$M_1 \otimes \ldots \otimes M_n$ is the initial object of $\tau_{(M_1, \ldots, M_n)}$. The Theorems 4.3 and 4.4 then lead to the expected result.

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