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BANDS OF MONOIDS

Stojan Bogdanović and Miroslav Ćirić

In this paper a construction of a band of monoids in general case is given.

A.H. Clifford, [1] gave a construction for a semilattice of groups by a transitive system of homomorphisms (called also strong semilattice of groups). B.M. Schein, [3] generalized Clifford's result to the band of unipotent monoids. For some related results see also [4, p.100]. One generalization of Clifford's result is given by M. Yamada, [5]. He considered a construction of a systematic normal band of P-monoids. In this paper we give a construction of a band of monoids in general case.

Let S be a band I of monoids S_i with identities e_i, $i \in I$, and let the following condition holds:

$$ij = i \Longrightarrow e_i e_j = e_i \wedge ij = j \Longrightarrow e_i e_j = e_j.$$

Then S is a systematic band I of monoids S_i, $i \in I$, [5].

For the notions and notations which are not defined here we refer to [2] and [4].

The main result of this paper is the following theorem.

Theorem 1. Let I be a band. To each $i \in I$ we associate a monoid S_i with the identity e_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Let $\leq_1$ and $\leq_2$ be quasiorders on I defined in the following way:

$$i \leq_1 j \Longleftrightarrow ji = i, \quad i \leq_2 j \Longleftrightarrow ij = i.$$

Let ϕ_{ij} and ψ_{ij} be homomorphisms of S_j into S_i over $\leq_1$ and $\leq_2$, respectively, for which the following properties hold:

(1) for every $i \in I$, ϕ_{ii} and ψ_{ii} are the identical automorphisms of S_i,

(2) $\phi_{ij} \circ \phi_{jk}(s_k) = \phi_{ik}(s_k)\phi_{ij}(e_j)$, $i \leq_1 j \leq_1 k$,

(3) $\psi_{ij} \circ \psi_{jk}(s_k) = \psi_{ij}(e_j)\psi_{ik}(s_k)$, $i \leq_2 j \leq_2 k$.

Let (a_{ij}) be an $(I\times I)$-matrix over $S = \bigcup_{i\in I} S_i$ such that $a_{ij}\in S_{ij}$, $a_{ii} = e_i$ and

(4) $\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k} = a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})$

for all $i,j,k\in I$. Define a multiplication $*$ on S by:

(5) $s_i * s_j = \phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j)$, $s_i\in S_i$, $s_j\in S_j$.

Then $(S,*)$ is a band of monoids.

Conversely, every band of monoids can be so constructed.

<u>Proof</u>. Let $s_i\in S_i$, $s_j\in S_j$ and $s_k\in S_k$. Then

$$
\begin{aligned}
(s_i*s_j)*s_k &= (\phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))*s_k \\
&= \phi_{ijk,ij}(\phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,ij}(\phi_{ij,i}(s_i))\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)\phi_{ijk,ij}(e_{ij})\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)\phi_{ijk,ij}(e_{ij}a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)(\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k})\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)(a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}))\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}e_{jk})\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})\psi_{ijk,jk}(e_{jk})\psi_{ijk,k}(s_k) \\
&= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})\psi_{ijk,jk}(\psi_{jk,k}(s_k)) \\
&= \phi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}\psi_{jk,k}(s_k)) \\
&= s_i*(\phi_{jk,j}(s_j)a_{jk}\psi_{jk,k}(s_k)) \\
&= s_i*(s_j*s_k).
\end{aligned}
$$

Hence, $(S,*)$ is a semigroup. It is clear that S is a band I of monoids S_i, $i\in I$.

Conversely, let S be a band I of monoids S_i, $i\in I$. By e_i we denote the identity of S_i, $i\in I$. We define mappings

$$\phi_{ij}(s_j) = s_je_i,\ i \leq_1 j,\quad \psi_{ij}(s_j) = e_is_j,\ i \leq_2 j.$$

Then

$$
\begin{aligned}
\phi_{ij}(s_jt_j) &= s_jt_je_i = s_jt_je_{ji},\ (\text{since } i = ji) \\
&= s_je_{ji}t_je_{ji} = s_je_it_je_i = \phi_{ij}(s_j)\phi_{ij}(t_j).
\end{aligned}
$$

Thus ϕ_{ij} is a homomorphism of S_j into S_i. Let $i \leq_1 j \leq_1 k$. Then

$$\phi_{ij}\circ\phi_{jk}(s_k) = \phi_{ij}(s_k e_j) = (s_k e_j)e_i = s_k(e_j e_i) =$$
$$= s_k(e_i(e_j e_i)) = (s_k e_i)(e_j e_i) =$$
$$= \phi_{ik}(s_k)\phi_{ij}(e_j).$$

Therefore, the condition (2) holds. In a similar way we prove that ψ_{ij} is a homomorphism and that (3) holds. It is clear that $a_{ij} = e_i e_j \in S_{ij}$ and that $a_{ii} = e_i$, for all $i,j\in I$. Now, we prove that the condition (4) holds. Indeed,

$$\phi_{ijk,ij}(a_{ij}\psi_{ij,j}(s_j))a_{ij,k} = \phi_{ijk,ij}(a_{ij}e_{ij}s_j)a_{ij,k} =$$
$$= a_{ij}e_{ij}s_j e_{ijk}a_{ij,k} = a_{ij}s_j a_{ij,k} = e_i e_j s_j e_{ij}e_k = e_i s_j e_k =$$
$$= e_i e_{jk}s_j e_k = a_{i,jk}s_j e_k = a_{i,jk}s_j e_j e_k = a_{i,jk}s_j a_{jk} =$$
$$= a_{i,jk}e_{ijk}s_j a_{jk} = a_{i,jk}e_{ijk}s_j e_{jk}a_{jk} = a_{i,jk}\psi_{ijk,jk}(s_j e_{jk}a_{jk}) =$$
$$= a_{i,jk}\psi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk}).$$

Finally,

$$s_i s_j = s_i e_i e_j s_j = s_i e_{ij}e_i e_j e_{ij}s_j = s_i e_{ij}a_{ij}e_{ij}s_j =$$
$$= \phi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j) \quad (=s_i * s_j). \quad ||$$

<u>Theorem</u> 2. Let I be a semilattice. To each $i\in I$ we associate a monoid S_i with the identity e_i such that $S_i \cap S_j = \emptyset$ if $i\neq j$. For each pair i,j of elements of I such that $i \leq j$, let $\phi_{ij}: S_j \to S_i$ be a homomorphism for which:

(6) ϕ_{ii} is the identical automorphism of S_i for each $i\in I$,

(7) $\phi_{ij}\circ\phi_{jk}(s_k) = \phi_{ik}(s_k)\phi_{ij}(e_j) = \phi_{ij}(e_j)\phi_{ik}(s_k)$,

for every $i,j,k\in I$ such that $i \leq j \leq k$. Let (a_{ij}) be an $(I\times I)$-matrix over $S = \bigcup_{i\in I}$ such that $a_{ij}\in S_{ij}$, $a_{ii} = e_i$ and

(8) $\phi_{ijk,ij}(a_{ij}\phi_{ij,j}(s_j))a_{ij,k} = a_{i,jk}\phi_{ijk,jk}(\phi_{jk,j}(s_j)a_{jk})$.

Define a multiplication $*$ on S by:

$$s_i * s_j = \phi_{ij,i}(s_i)a_{ij}\phi_{ij,j}(s_j), \quad s_i\in S_i,\ s_j\in S_j).$$

Then $(S,*)$ is a semilattice of monoids.

Conversely, every semilattice of monoids can be constructed.

Proof. Let S be a semilattice I of monoids S_i and let e_i be the identity of S_i, $i\in I$. For $i,j\in I$, $i \le j$ we define the following mappings:

$$\phi_{ij}(s_j) = s_j e_i, \quad \psi_{ij}(s_j) = e_i s_j.$$

Then

$$\phi_{ij}(s_j) = \phi_{ij,j}(s_j) = s_j e_{ij} = e_{ij} s_j e_{ij} = $$
$$= e_{ij} s_j = \psi_{ij,j}(s_j) = \psi_{ij}(s_j)$$

if $i \le j$. Thus

(9) $\phi_{ij}(s_j) = \psi_{ij}(s_j), \quad i \le j.$

By Theorem 1. we have that the conditions (2) and (3) hold and by (9) it follows that

$$\phi_{ik}(s_k)\phi_{ij}(e_j) = \phi_{ij}\circ\phi_{jk}(s_k) = \psi_{ij}\circ\psi_{jk}(s_k) =$$
$$= \psi_{ij}(e_j)\psi_{ik}(s_k) = \phi_{ij}(e_j)\phi_{ik}(s_k).$$

Therefore, the condition (7) holds. The remains of the proof follows by Theorem 1. and by (9).

The converse is similar as in the proof of Theorem 1. ||

Theorem 3. Let I be a band. To each $i\in I$ we associate a monoid S_i with the identity e_i such that $S_i \cap S_j = \emptyset$ if $i\neq j$. Let ϕ_{ij} and ψ_{ij} be homomorphisms for which the conditions (1)-(4) holds and

$$\phi_{ij}(e_j) = e_i, \quad \psi_{ij}(e_j) = e_i.$$

Let (a_{ij}) be an $(I\times I)$-matrix over $S = \bigcup_{i\in I} S_i$ such that $a_{ij} \in S_{ij}$, $a_{ii} = e_i$ and

$$ij = i \Longrightarrow a_{ij} = a_{ii} \wedge ij = j \Longrightarrow a_{ij} = a_{jj}.$$

Define a multiplication $*$ on S by (5). Then $(S,*)$ is a systematic band.

Conversely, every systematic band can be so constructed. ||

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TRAKE MONOIDA

S. Bogdanović, M. Ćirić

R e z i m e

U radu je dat opis trake monoida u opštem slučaju.

Faculty of Economics
18000 Niš
Trg JNA 11
Yugoslavia

Faculty of Mechanical Engineering
18000 Niš
Beogradska 14
Yugoslavia