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ON CONVOLUTIONS AND NEUTRIX CONVOLUTIONS INVOLVING SLOWLY VARYING FUNCTIONS

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Abstract

Let $L(x)$ be a slowly varying function at both zero and infinity. The existence of the commutative neutrix convolution product of the distributions $L(x)_-$ and x_+^r where $r = 0, 1, 2...$ is proved.

1. Introduction

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined as follows, see [3].

Definition 1. Let f and g be distributions in $\mathcal{D}'$, satisfying at least one of the following conditions:

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Then the *convolution product* $f * g$ is defined by

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle$$

for arbitrary φ in $\mathcal{D}$.

It follows that if the convolution product $f * g$ exists by this definition then

$$f * g = g * f, \tag{1}$$

$$(f * g)' = f * g' = f' * g. \tag{2}$$

The convolution product of distributions may be defined in a more general way yet, without the restrictions on the supports given above in

a) or b). However, the convolution product in the sense of any of these definitions does not exist for many pairs of distributions. In [4] (and other papers) the *commutative neutrix convolution product* was defined so it exists for a considerable larger class of pairs of distributions. In that definition unit sequences of functions in $\mathcal{D}$ are used, which allows one to approximate a given distribution by a sequence of distribution of bounded support.

To recall the definition of the commutative neutrix convolution product we first of all let τ be a fixed function in $\mathcal{D}$ with the following properties:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \le \tau(x) \le 1$,

(iii) $\tau(x) = 1, \text{ for } |x| \le \frac{1}{2}$,

(iv) $\tau(x) = 0, \text{ for } |x| \ge 1$.

Next we define the unit sequence $(\tau_n)_{n\in\mathcal{N}}$ of functions, setting

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x + n^{n+1}), & x > n, \\ \tau(n^n x - n^{n+1}), & x < -n. \end{cases}$$

In order to define the commutative neutrix convolution product, we need the following definition given by Van der Corput (see [2]):

Definition 2. A *neutrix* N is a commutative additive group of functions $\nu : N' \to N''$ (where the domain N' is a set and the range N'' is a commutative additive group) with the property that if ν is in N and $\nu(\xi) = \gamma$ for all ξ in N' then $\gamma = 0$. The functions in N are said to be negligible.

Example 1. Let N' be the closed interval $[0,1] = \{x : 0 \le x \le 1\}$ and let N be the set of all functions defined on N' of the form

$$a \sin x + bx^2,$$

where a and b are arbitrary real numbers. Then N is a neutrix, since if

$$a \sin x + bx^2 = c$$

for all x in N', then $a = b = c = 0$.

Now suppose that N' is contained in a topological space with a limit point b which is not in N', and let N be a commutative additive group of functions $\nu : N' \to N''$ with the property that if N contains a function of ξ which tends to a finite limit γ as ξ tends to b, then $\gamma = 0$. It follows that N is a neutrix. If now $f : N' \to N''$ and there exists a constant β such that $f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit of $f(\xi)$ as ξ tends to b and we write $\text{N}-\lim_{\xi\to b} f(\xi) = \beta$. Note that if a neutrix limit exists, then it is unique, since if $f(\xi) = \beta$ and $f(\xi) = \beta'$ are in N, then the constant function $\beta - \beta'$ is also in N and so $\beta = \beta'$.

In the following we let N the neutrix having domain $N' = \mathcal{N} = \{1, 2, \ldots, n, \ldots\}$, range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \ \ln^{r} n, \quad (\lambda \neq 0, \ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Also, we need the following definition given by Van der Corput (see [2]):

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \ldots$. Then the *commutative neutrix convolution product* $f \boxed{*} g$ is defined as the neutrix limit of the sequence $(f_n * g_n)_{n \in \mathbf{N}}$, provided that the limit h exists in the sense that

$$N\text{-}\lim_{n \to \infty} \langle f_n * g_n, \varphi \rangle = \langle h, \varphi \rangle,$$

for all φ in $\mathcal{D}$, where N is the neutrix described in Definition 2.

Note that in this definition the convolution product $f_n * g_n$ is in the sense of Definition 1. Namely, the distributions f_n and g_n have bounded support since the support of τ_n is contained in the interval $[-n - n^{-n}, n + n^{-n}]$. From the following theorem, proved in [4] (see also [5],) it follows that the commutative neutrix convolution product from Definition 3 is a proper generalization of the "classical" convolution product of distributions from Definition 1.

Theorem 1. Let f and g be distribution in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Definition 1. Then the commutative neutrix convolution product $f \boxed{*} g$ exists and

$$f \boxed{*} g = f * g.$$

On using Definition 3, one can find several important neutrix products of distributions, see [5].

Equation (2) does not necessarily hold, since $(f \boxed{*} g)'$ is not necessary equal to $f' \boxed{*} g$. In fact, we have following lemma which was also proved in [4].

Lemma 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative neutrix convolution product $f \boxed{*} g$ exists. If $N\text{-}\lim_{n \to \infty} \langle (f\tau_n') * g_n, \varphi \rangle$ exists and equals $\langle h, \varphi \rangle$ for all φ in $\mathcal{D}$, then $f' \boxed{*} g$ exists and*

$$(f \boxed{*} g)' = f' \boxed{*} g + h. \tag{3}$$

Here and also throughout this paper $L : (0, \infty) \to (0, \infty)$ is a given locally integrable function which satisfied the following conditions

$$\lim_{x\to 0^+}\frac{L(kx)}{L(x)}=1 \qquad \text{for any} \quad k>0, \tag{4}$$

$$\lim_{x\to\infty}\frac{L(kx)}{L(x)}=1 \qquad \text{for any} \quad k>0. \tag{5}$$

A positive locally integrable function satisfying (4),(resp.(5)) is called *slowly varying at zero (slowly varying at infinity)*. The first example of a function satisfying the relations (4) and (5) is the logarithm; other examples are the positive powers and the iterations of the logarithm, e.g., $\ln^3$ and $\ln\ln$. An exposition of the theory of slowly varying functions can be found in [8].

The distribution $L(x)_-$ is defined by:

$$L(x)_- = \begin{cases} L(-x) & x<0 \\ 0 & x>0 \end{cases}$$

The aim of this paper is to analyze neutrix product involving slowly varying functions. For that reason we now take the set of negligible functions obtained by replacing the logarithmic function ln with the slowly varying function L. More precisely, our new neutrix, again denoted by N, will have the domain $N'=\{1,2,\ldots,n,\ldots\}$, the range of the real numbers $\mathbf{R}$, with negligible functions finite linear sums of the functions

$$n^\lambda,\ n^\lambda L(n),\ L^r(n)$$

for all real $\lambda \neq 0$ and $r \in \mathbf{N}$ (instead of the functions from (3)), and all functions which converge to zero in the usual sense, as n tends to infinity. In this way, we obtain a wide range of neutrix products, involving the corresponding slowly varying function.

2. Main theorem

Before we turn to the announced neutrix products, we cite two statements that we need later on.

Lemma 1. *Let $L(x)$ be a slowly varying function at infinity. Then $K(x)=L(\frac{1}{x})$ is slowly varying function at zero.*

Theorem 2.*Let L be a slowly varying function at infinity and let f be a locally integrable function on the interval $[a,b]$ with the property that*

$$\int_a^b x^\delta |f(x)|\,dx < \infty \qquad \text{for some} \quad \delta>0.$$

Then the integral

$$\Phi(t) = \int_a^b f(x)L(tx)\,dx$$

exists and

$$\Phi(t) \sim L(t) \int_a^b f(x)\,dx \qquad \text{as} \quad t \to +\infty.$$

We now state and prove the following statement, which proves the existence of a commutative neutrix convolution product involving a slowly varying function.

Theorem 4. *Let L be a slowly varying function both at zero and at infinity. Then commutative neutrix convolution product $x_-^\lambda L(x)_- \boxed{*} x_+^\mu$ exists and*

$$x_-^\lambda L(x)_- \boxed{*} x_+^\mu = 0$$

for

$$\lambda, \mu \neq -1, -2, \dots \quad \text{and} \quad \lambda + \mu \neq 0, \pm 1, \pm 2, \dots.$$

Proof:

We will suppose first of all that $\lambda, \mu > -1$ and $\lambda + \mu \neq -1, 0, 1, 2, \dots$ so that $x_-^\lambda L(x)_-$ and x_+^μ are locally summable functions. Put

$$\left[x_-^\lambda L(x)_-\right]_n = x_-^\lambda L(x)_- \tau_n(x)\,, \qquad \left[x_+^\mu\right]_n = x_+^\mu \tau_n(x).$$

Then the convolution product $x_-^\lambda L(x)_- * x_+^\mu$ exists by Definition 1 in the classical case and so:

$$\begin{aligned} \left[x_-^\lambda L(x)_-\right]_n * \left[x_+^\mu\right]_n &= \int_{-\infty}^{\infty} \left[t_-^\lambda L(t)_-\right]_n \left[(x-t)_+^\mu\right]_n dt \\ &= \int_{-\infty}^{\infty} t_-^\lambda L(t)_- (x-t)_+^\mu \tau_n(t)\tau_n(x-t)\,dt. \end{aligned} \tag{7}$$

For $-n \leq x \leq 0$ it holds

$$\int_{-\infty}^{\infty} t_-^\lambda L(t)_- (x-t)_+^\mu \tau_n(t)\tau_n(x-t)\,dt = \int_{-n}^{x} (-t)^\lambda L(-t)(x-t)^\mu\,dt$$

$$+ \int_{-n-n^{-n}}^{-n} (-t)^\lambda L(-t)(x-t)^\mu \tau_n(t)\tau_n(x-t)\,dt$$

$$= I_1 + I_2.$$

Making the substitution $t = -xnu$, we have:

$$\int_{-n}^{x} (-t)^\lambda L(-t)(x-t)^\mu \, dt = x^{\lambda+\mu+1} n^{\lambda+1} \int_{-\frac{1}{n}}^{\frac{1}{x}} u^\lambda (nu+1)^\mu L(xnu) \, du.$$

On using Theorem 2 we can see that the right hand side behaves as:

$$L(xn)\,(1+nu)^\mu \left(n^{-1}+x^{-1}\right) .$$

Now we have:

$$N-\lim_{n\to\infty} I_1 = 0. \tag{8}$$

When $n \geq x \geq 0$, we have

$$\int_{-\infty}^{\infty} t_-^\lambda L(t)_-(x-t)_+^\mu \tau_n(t)\tau_n(x-t)\, dt = \int_{x-n}^{0} (-t)^\lambda L(-t)(x-t)^\mu \, dt$$

$$+ \int_{-n-n^{-n}}^{-n} (-t)^\lambda L(-t)(x-t)^\mu \tau_n(t)\tau_n(x-t)\, dt.$$

Making the substitution $t = -xn(1+u)$, we have:

$$\int_{x-n}^{0} (-t)^\lambda L(-t)(x-t)^\mu \, dt =$$

$$= n^{\lambda+1} x^{\lambda+\mu+1} \int_{1-\frac{1}{n}\frac{1}{x}}^{1} (1-u)^\lambda (n+nu+1)^\mu L\left(xn(1+u)\right) du.$$

It follows as above that

$$N-\lim_{n\to\infty} \int_{x+n}^{0} (-t)^\lambda L(-t)(x-t)^\mu \, dt = 0. \tag{9}$$

Further, it is easily seen that

$$\int_{-n-n^{-n}}^{-n} (-t)^\lambda L(-t)(x-t)^\mu \tau_n(t)\tau_n(x-t) dt = O(n^{-n+\lambda+\mu} L(n))$$

and so

$$\lim_{n\to\infty} \int\limits_{-n-n^{-n}}^{-n} (-t)^\lambda L(-t)(x-t)^\mu \tau_n(t)\tau_n(x-t)dt = 0. \tag{10}$$

Now it follows from the equations (7) , (8) , (9) and (10) that the commutative neutrix convolution product $x_-^\lambda L(x)_- \boxed{*} x_+^\mu$ exists and it is equal to zero, proving the theorem for the case $\lambda, \mu > -1$ and $\lambda+\mu \neq -1, 0, 1, 2, \ldots$.

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КОНВОЛУЦИИ И НЕУТРИКС КОНВОЛУЦИИ СО СПОРО ПРОМЕНЛИВИ ФУНКЦИИ

Биљана Јолевска–Тунеска

Р е з и м е

Нека $L(x)$ е споро променлива функција во нула и во бесконечност. Да дени се потребни и доволну услови за постоење на комутативниот неутрикс конволуциски производ на дистрибуциите $L(x)_-$ и x_+^r каде што $r = 0, 1, 2...$

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