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ANALYTIC REPRESENTATION OF DISTRIBUTION AS AN ANALYTIC CONTINUATION OF A POWER SERIES

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Abstract

In this work we give a theorem of aplying the theory of distributions in the theory of analytic functions. The theorem is about the analytic continuation of power series. We give also an example as a illustration.

The analytic representation of a distribution is an important operation in the theory of distribution, above all, having its application in the study of distributions as limits to the analytic functions in certain domain. Further on, with the help of the analytic representation in specific cases, we can easily determine their Fourier or Laplace transform, and on the other hand, analytic representation can be used in the study of complex functions.

With this work, we give a small contribution to the use of distributions in the theory of analytic functions.

The used symbols are customary to the theory of distributions, or more specifically, in our case, they are the same as in (1).

It is known that for every distribution $T \in D'$, where D' is the space of the Schwartz distributions on the real straight line R, there is a complex function $f(z)$, $z = x + iy$ that is analytic for $y \neq 0$ and in that $f(x+iy) - f(x-iy) \to T$ when $y \to 0^+$ in terms of distributions. The function $f(z)$ is called analytic representation of the distribution T. Actually, the function $f(z)$ is analytic to the complex plane C, except to the support "supp$T$" of the distribution (1.p.76).

The determination of an analytic representation for a given distribution T, in a general case is not easy. In (1. p. 81 and further on), spaces O_α are given, especiely adapted for the determination of the analytic representation of the distributions. We present the following theorem.

Theorem. Let $g(t)$ be a continuous function in the interval $(1, \infty)$ and equal to 0 for $t \leq 1$. We assume that it is locally integrable and satisfies the condition $|g(t)| = O(|t|^\beta)$, $\beta < 0$. In that case, its Cauchy representation $\hat{g}(z)$ is an analytic continuation of a power series whose coefficients are determined with $g(t)$.

Proof. From the condition $\alpha + \beta + 1 < 0$ (1. p. 82) it follows that $g(t)$ determines a distribution of the space O_α for $\alpha + \beta < -1$. Since $\beta < 0$, $g(t) \in O'_{-1}$ and that the function

$$\hat{g}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)\,dt}{t - z}, \quad z = x + iy, \quad y \neq 0 \tag{1}$$

is analytic in the domain $\Omega = C \backslash [1, \infty)$, and it is an analytic representation for the distribution $g(t)$.

This means that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \hat{g}(x + i\varepsilon) - \hat{g}(x - i\varepsilon)]\varphi(x)dx = \int_{1}^{\infty} g(x)\varphi(xdx, \quad \varphi \in D.$$

D is the space of the test functions of R. If in (1) we put $t = e^u$ we get:

$$\hat{g}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{g(e^u)e^u\,du}{e^u - z} = \frac{1}{2\pi i} \int_0^\infty \frac{g(e^u)du}{1 - ze^{-u}}.$$

For $|z| < 1$ and $|ze^{-u}| < 1$ we get

$$\begin{aligned}
\hat{g}(z) &= \frac{1}{2\pi i} \int_0^\infty g(e^u) \sum_{n=0}^{\infty} z^n e^{-nu} du \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i} z^n \int_0^\infty g(e^u) e^{-nu} du \\
&= \sum_{n-0}^{\infty} a_n z^n,
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_0^\infty g(e^u)e^{-nu}du. \tag{3}$$

Since $\hat{g}(z)$ is analytic in $\Omega = C\backslash[1,\infty)$ it follows that the radius of the convergence r for the power series is $r = 1$, and the function $\hat{g}(z)$ is an analytic continuation of the power series of the domain Ω.

Example. Let $g(t) = \sin\frac{1}{t}$ for $t > 1$ and 0 for $t \le 1$ than

$$a_n = \frac{1}{2\pi i}\int_0^\infty \sin\frac{1}{e^u}(e^{-u})^n du$$

for change $t = e^{-u}$

$$a_n = \frac{1}{2\pi i}\int_0^1 \sin t \cdot t^{n+1}\, dt.$$

Note. In some examples we have the opposite task: Let the power series $\sum_{n=0}^{\infty} a_n z^n$ have a convergence radius $r = 1$, and in that

$$a_n = \int_0^1 g(t)t^n\, dt,$$

$g(t)$ is a continuous function. By replacement in the series we get

$$f(z) = \sum_{n=0}^{\infty}\Big[\int_0^1 g(t)t^n\, dt\Big]z^n = \sum_{n=0}^{\infty}\int_0^1 g(t)(tz)^n\, dt, \quad |z| < 1$$

$$f(z) = \int_0^1 g(t)\sum_{n-0}^{\infty}(tz)^n\, dt = \int_0^1 g(t)\frac{1}{1-tz}\, dt.$$

By changing $t = \frac{1}{u}$ we get

$$f(z) = \int_1^\infty \frac{\frac{1}{u}g\left(\frac{1}{u}\right)du}{u-z}.$$

The distribution

$$T = \left[\frac{1}{t} g\left(\frac{1}{t}\right) H(t-1)\right],$$

$H(t)$ is the Heaviside function, has a Cauchy representation $\hat{T}(z)$ equal to

$$\hat{T}(z) = \frac{1}{2\pi i} f(z).$$

According to this the function $f(z)$ is analytic in the domain $\Omega = C\backslash[1,\infty)$ with a jump for $x > 1$ starting from the upper half plane towards the lower one, equal to $\frac{1}{x} g\left(\frac{1}{x}\right)$. In that way we get the function $f(z) = 2\pi i \hat{T}(z)$ which is an analytic continuation for the given power series.

Example. To determine the analytic continuation of the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n^p}, \qquad p > 0,$$

we put

$$\frac{1}{n^p} = c_p \int_0^{\infty} e^{-nt} t^{p-1}\, dt.$$

The constant c_p should be determined. By setting $u = nt$ we get

$$\frac{1}{n^p} = c_p \cdot \frac{1}{n^p} \int_0^{\infty} u^{p-1} e^{-u} du = c_p \cdot \frac{1}{n^p} \Gamma(p),$$

$\Gamma(p)$ is the gama function. $c_p = \dfrac{1}{\Gamma(p)}$. By setting in the power series we get

$$\sum_{n=0}^{\infty} z^n \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-nt} t^{p-1}\, dt = \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-1} \sum_{n=0}^{\infty} (ze^{-t})^n\, dt$$

$$= \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-1} \frac{1}{1 - ze^{-t}}\, dt$$

$$= \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{t^{p-1} e^t}{e^t - z}\, dt.$$

If we put $u = e^t$ we get

$$\sum_{n=0} \frac{z^n}{n^p} = \frac{1}{\Gamma(p)} \int_1^\infty \frac{(\ln u)^{p-1} \frac{1}{u} du}{u - z} \quad \text{for} \quad |z| < 1.$$

The distribution

$$T = \frac{1}{\Gamma(p)} \left[(\ln t)^{p-1} \cdot \frac{1}{t} H(t-1) \right]$$

has Cauchy representation

$$\hat{T}(z) = \frac{1}{2\pi i} f(z) \quad \text{where} \quad f(z) = \frac{1}{\Gamma(p)} \int_1^\infty \frac{(\ln t)^{p-1} \frac{1}{t}}{t - z} dt.$$

By analogy, the function $f(z)$ is analytic to the domain $\Omega = C \backslash [1, \infty)$ with a jump for $x > 1$ equal to $\dfrac{1}{\Gamma(p)} \dfrac{(\ln x)^{p-1}}{x}$ and it represents analytic continuation of the power series.

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АНАЛИТИЧКА РЕПРЕЗЕНТАЦИЈА НА ДИСТРИБУЦИИ КАКО АНАЛИТИЧКО ПРОДОЛЖУВАЊЕ НА СТЕПЕНСКИ РЕД

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Резиме

Во оваа работа дадена е една теорема за примената на дистрибуциите во теоријата на аналитичните функции. Теоремата се однесува на аналитичното продолжување на степенски ред. Даден е исто така еден пример како илустрација

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