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DUAL SPACE OF THE SPACE OF BOUNDED LINEAR n-FUNCTIONALS

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Abstract. In [2] are considered n-Banach spaces, and in [4] are considered bounded and continuous linear n-functionals defined on n-normed space and several theorems connected with them, are proved. Then is proved that: Linear n-functional F is continuous if and only if F is bounded (theorem 4). In this paper, a dual space X^* of space of bounded linear n-functionals is considered and it is proved that: if X is n-Banach space than $(X*, ||.||)$ is Banach space.

1. Introduction

Definition 1. *Let $X_i,\ i = 1, 2, \dots, n$ be linear subspace of same vector nnormed space. Then the mapping $F : X_1 \times \dots \times X_n \to \mathbb{R}$ is called* **n-functional** *with domain $X_1 \times X_2 \times \dots \times X_n$.*

Definition 2. *Let F be nfunctional with domain $X_1 \times X_2 \times \dots \times X_n$. Then F is* **linear** $n-$ **functional** *if the following conditions are satisfied:*

1. $F(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = \sum\limits_{\substack{z_i \in \{x_i, y_i\} \\ i=1,\dots,n}} F(z_1, z_2, \dots, z_n)$
2. $F(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) = \alpha_1 \alpha_2 \dots \alpha_n F(x_1, x_2, \dots, x_n)$
$\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, n$

Definition 3. *Let X be n-normed space. Let F be n-functional with domain $D(F) \subseteq X^n$ then F is bounded if there exists real number $K \geqslant 0$ such that*
$F(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) = \alpha_1 \alpha_2 \dots \alpha_n F(x_1, x_2, \dots, x_n)$.

Let F be bounded n-functional, we define **norm** of F, denoted by $||F||$, with

$$||F|| = \inf \{K \, ||F(x_1, x_2, \dots, x_n)| \leqslant K||x_1, x_2, \dots, x_n||, (x_1, x_2, \dots, x_n) \in D(F)\} \quad (1)$$

If F is unbounded n-functional, then we define $||F|| = +\infty$.

In this context for bounded linear n-functionals in [4] the following properties are proved.

Lemma 1. *Let F be a bounded linear n-functional and x_i, $i = 1, \dots, n$, are linearly dependent vectors such that $(x_1, x_2, \dots, x_n) \in D(F)$. Then $F(x_1, x_2, \dots, x_n) = 0$.*

Theorem 1. *Let F be a bounded linear n-functional on domain $D(F)$. Then*

$$||F|| = \sup\{|F(x_1, x_2, \dots, x_n)|; ||x_1, x_2, \dots, x_n|| = 1, (x_1, x_2, \dots, x_n) \in D(F)\}$$
$$= \sup\left\{\frac{|F(x_1,x_2,\dots,x_n)|}{||x_1,x_2,\dots,x_n||}; ||x_1, x_2, \dots, x_n|| \neq 0, (x_1, x_2, \dots, x_n) \in D(F)\right\}.$$

Further on, continuity of linear n-functional is defined as following.

Definition 4. *Let F be n-functional. Then F is* **continuous at the point** *$(x_1, x_2, \dots, x_n)$ if for all $\varepsilon > 0$ exist $\delta > 0$ such that*

$$|F(x_1, x_2, \dots, x_n) - F(y_1, y_2, \dots, y_n)| < \varepsilon$$

always when

$$||z_{1j}, z_{2j}, \dots, z_{nj}|| < \delta$$

where

$$z_{ij} = \begin{cases} x_i - y_i, i = j \\ x_i \vee y_i, i \neq j \end{cases}$$

for $j = 1, 2, \dots, n$. The n-functional F is continuous if F is continuous at every point from its domain.

In [4], for continuous n-functionals are proved the following properties.

Theorem 2. *If the linear n-functional F is continuous at the point $(0, 0, \dots, 0)$, then F is continuous at every point from its domain $D(F)$.*

Theorem 3. *Linear n-functional F is continuous if and only if F is bounded.*

Definition 5. *The sequence $\{x_k\}$ from the vector n-normed space L is Cauchy sequence if there exists linear independent vectors $y_1, y_2, ..., y_n$ such that*

$$\lim_{k,m\to\infty} ||x_k - x_m, y_2, ..., y_{n-1}, y_n|| = 0$$
$$\lim_{k,m\to\infty} ||x_k - x_m, y_1, ..., y_{i-1}, y_{i+1}, ..., y_n|| = 0, \quad i = 2, ..., n-1$$
$$\lim_{k,m\to\infty} ||x_k - x_m, y_1, ..., y_{n-1}|| = 0.$$

Definition 6. *The sequence* $\{ x_k \}$ *from n-normed space L is convergent if there exist* $x \in L$ *such that*
$\lim_{k\to\infty} \|x_k - x, y_1, ..., y_{n-1}\| = 0$, *for all* $y_1, y_2, ..., y_{n-1} \in L$.

For x we shall say that is limit for the sequence $\{ x_k \}$ and we'll write $x_k \to x,\ k \to \infty$.

Definition 7. *For n-normed space L, well say that is n-Banach space if every Cauchy sequence is convergent.*

In [4] the following property is proved.

Theorem 4. *Every real n-normed vector space with dimension n is n-Banach space.*

2. DUAL SPACE OF THE SPACE OF BOUNDED LINEAR n-FUCTIONALS

Definition 8. *Let X be* $n-$*Banach space,* X^* *is a set of bounded linear n-functionals on domain* X^n *and let* $F, G \in X^*$. *We define*

a) $F = G$ *if* $F(x_1, x_2, \ldots, x_n) = G(x_1, x_2, \ldots, x_n)$, *for all* $(x_1, x_2, \ldots, x_n) \in X^n$,

b) $(F+G)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)$, *for all* $(x_1, x_2, \ldots, x_n) \in X^n$,

c) $(\alpha F)(x_1, x_2, \ldots, x_n) = \alpha F(x_1, x_2, \ldots, x_n)$, *for all* α *and all* $(x_1, x_2, \ldots, x_n) \in X^n$.

Theorem 5. *Let X be n-Banach space. Then* $(X^*, ||.||)$ *is Banach space.*

Proof. Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ and $\alpha_i \in \mathbb{R}, i = 1, 2, \ldots, n$. Then according to Definition 2. we have

$$
\begin{aligned}
&(F+G)(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = \\
&= F(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + G(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = \\
&= \sum_{\substack{z_i \in \{x_i, y_i\} \\ i=1,2,\ldots,n}} F(z_1, z_2, \ldots, z_n) + \sum_{\substack{z_i \in \{x_i, y_i\} \\ i=1,2,\ldots,n}} G(z_1, z_2, \ldots, z_n) = \\
&= \sum_{\substack{z_i \in \{x_i, y_i\} \\ i=1,2,\ldots,n}} (F+G)(z_1, z_2, \ldots, z_n)
\end{aligned}
$$

$$
\begin{aligned}
&(F+G)(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \\
&\qquad = F(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) + G(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) \\
&\qquad = \alpha_1 \alpha_2 \ldots \alpha_n F(x_1, x_2, \ldots, x_n) + \alpha_1 \alpha_2 \ldots \alpha_n G(x_1, x_2, \ldots, x_n) \\
&\qquad = \alpha_1 \alpha_2 \ldots \alpha_n [F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)] \\
&\qquad = \alpha_1 \alpha_2 \ldots \alpha_n (F+G)(x_1, x_2, \ldots, x_n).
\end{aligned}
$$

Further on, because of Definition 3 we have

$$\begin{aligned}|(F+G)(x_1,x_2,\dots,x_n)| &= |F(x_1,x_2,\dots,x_n)+G(x_1,x_2,\dots,x_n)|\\ &\le |F(x_1,x_2,\dots,x_n)|+|G(x_1,x_2,\dots,x_n)|\\ &\le ||F||\cdot||x_1,x_2,\dots,x_n||+||G||\cdot||x_1,x_2,\dots,x_n||\\ &= (||F||+||G||)||x_1,x_2,\dots,x_n||,\end{aligned}$$

which means that $F+G\in X^*$ and clearly $||F+G||\le||F||+||G||$.

Analogously we can prove that for every α and every $F\in X^*$, $\alpha F\in X^*$ and $||\alpha F||=|\alpha|\cdot||F||$ holds.

From the other hand, according to Definition 3 we have $|F(x_1,x_2,\dots,x_n)|\le||F||\cdot||x_1,x_2,\dots,x_n||$, for all $(x_1,x_2,\dots,x_n)\in X^n$, so $||F||=0$ if and only if $F=0$, which means that X^* is vector space with norm defined by (1).

Let $\{F_k\}$ be Cauchy sequence on $X*$, i.e. let

$$\lim_{\substack{m\to\infty\\ k\to\infty}}||F_k-F_m||=0 \tag{2}$$

Then for all $(x_1,x_2,\dots,x_n)\in X^n$ is true that

$$|F_k(x_1,x_2,\dots,x_n)-F_m(x_1,x_2,\dots,x_n)|\le||F_k-F_m||\cdot||x_1,x_2,\dots,x_n||$$

which means that for every $(x_1,x_2,\dots,x_n)\in X^n$ the real sequence $\{F_k(x_1,x_2,\dots,x_n)\}$ is a Cauchy sequence. On X^n let define functional F with

$F(x_1,x_2,\dots,x_n)=\lim\limits_{k\to\infty}F_k(x_1,x_2,\dots,x_n)$, $(x_1,x_2,\dots,x_n)\in X^n$.

Then, for all $(x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\in X^n$ and $\alpha_i\in\mathbb{R}, i=1,2,\dots,n$ we have

$$\begin{aligned}&F(x_1+y_1,x_2+y_2,\dots,x_n+y_n)=\lim_{k\to\infty}F_k(x_1+y_1,x_2+y_2,\dots,x_n+y_n)\\ &=\lim_{k\to\infty}\sum_{\substack{z_i\in\{x_i,y_i\}\\ i=1,\dots,n}}F_k(z_1,z_2,\dots,z_n)=\sum_{\substack{z_i\in\{x_i,y_i\}\\ i=1,\dots,n}}\lim_{k\to\infty}F_k(z_1,z_2,\dots,z_n)=\\ &=\sum_{\substack{z_i\in\{x_i,y_i\}\\ i=1,\dots,n}}F(z_1,z_2,\dots,z_n)\end{aligned}$$

and

$$\begin{aligned}F(\alpha_1x_1,\alpha_2x_2,\dots,\alpha_nx_n)&=\lim_{k\to\infty}F_k(\alpha_1x_1,\alpha_2x_2,\dots,\alpha_nx_n)\\ &=\lim_{k\to\infty}\alpha_1\alpha_2\dots\alpha_nF_k(x_1,x_2,\dots,x_n)\\ &=\alpha_1\alpha_2\dots\alpha_n\lim_{k\to\infty}F_k(x_1,x_2,\dots,x_n)\\ &=\alpha_1\alpha_2\dots\alpha_nF(x_1,x_2,\dots,x_n),\end{aligned}$$

i.e. F is $n$linear functional. On the other hand, for the sequence $\{F_k\}$, $|\,||F_k|| - ||F_m||\,| \leq ||F_k - F_m||$ holds.

Now from (2) we get that $\{||F_k||\}$ is real Cauchy sequence, which means that there exist $K \in \mathbb{R}$ such that $||F_k|| \leqslant K$, for all $k \in \mathbb{N}$, from where we get

$$\begin{aligned}|F(x_1, x_2, \ldots, x_n)| &= |\limsup_{k\to\infty} F_k(x_1, x_2, \ldots, x_n)| \\ &= \limsup_{k\to\infty} |F_k(x_1, x_2, \ldots, x_n)| \\ &\leq \limsup_{k\to\infty} ||F_k|| \cdot \ ||x_1, x_2, \ldots, x_n|| \\ &\leq K||x_1, x_2, \ldots, x_n||,\end{aligned}$$

i.e. $F \in X*$.

We'll prove that $\{F_k\}$ converges to F. Let $||x_1, x_2, \ldots, x_n|| \neq 0$. If $\varepsilon > 0$ is chosen, then from (2) we have that there exist $n_0 \in \mathbb{N}$ such that $||F_m - F_k|| < \varepsilon$ when $m, k > n_0$, so by Definition 3 we have

$$\begin{aligned}|F_m(x_1, x_2, \ldots, x_n) - F_k(x_1, x_2, \ldots, x_n)| &\leq ||F_m - F_k|| \cdot ||x_1, x_2, \ldots, x_n|| \\ &\leq \varepsilon||x_1, x_2, \ldots, x_n||,\end{aligned}$$

for all $m, k \geq n_0$. On the other hand, because of

$$F(x_1, x_2, \ldots, x_n) = \lim_{k\to\infty} F_k(x_1, x_2, \ldots, x_n)$$

there exist $M = M(x_1, x_2, \ldots, x_n) > n_0$ such that

$$|F_M(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| < \varepsilon||x_1, x_2, \ldots, x_n||.$$

So we have

$$\begin{aligned}&|F_k(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \leq \\ &\leq |F_k(x_1, x_2, \ldots, x_n) - F_M(x_1, x_2, \ldots, x_n)| + \\ &\quad + |F_M(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \\ &\leq \varepsilon||x_1, x_2, \ldots, x_n|| + \varepsilon||x_1, x_2, \ldots, x_n|| = 2 \cdot \varepsilon||x_1, x_2, \ldots, x_n||\end{aligned}$$

for $k > n_0$. If $||x_1, x_2, \ldots, x_n|| = 0$, then the vectors $x_1, x_2, \ldots, x_n$ are linearly dependent, and according to Lema 1 it follows that

$$F_k(x_1, x_2, \ldots, x_n) = 0 = F(x_1, x_2, \ldots, x_n)$$

which means $|F_k(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \leq 2 \cdot \varepsilon||x_1, x_2, \ldots, x_n||$, for all $k > n_0$. Hence, for all $(x_1, x_2, \ldots, x_n) \in X^n$ the following holds $|F_k(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \leq 2 \cdot \varepsilon||x_1, x_2, \ldots, x_n||$, for all $k > n_0$. i.e. accordingly to Definition 3 we get $||F_k - F|| \leq 2\varepsilon$, for $k > n_0$, i.e. $\{F_k\}$ converge to F.

Finally from the arbitrarily of the Cauchy sequence $\{F_k\}$ we have that $(X*, ||.||)$ is Banach space.

□

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ДУАЛЕН ПРОСТОР НА ПРОСТОРОТ ОГРАНИЧЕНИ ЛИНЕАРНИ n- ФУНКЦИОНАЛИ

Ристо Малчески, Здравко Цветковски

Р е з и м е

Во [2] се разгледани n-банаховите простори, а во [4] се разгледани ограничените и непрекинатите линеарни n-функционали дефинирани на n-нормиран простор и се докажани неколку тврдења во врска со истите. Притоа, е докажано дека: Линеарниот n-функционал F е непрекинат ако и само ако е ограничен (теорема 4). Во оваа работа е разгледан дуалниот простор X^* на просторот ограничени линеарни n-функционали и е докажано дека ако X е $n-$банахов простор, тогаш $(X^*, ||\cdot||)$ е Банахов простор.

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