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SOME GRÜSS TYPE INTEGRAL INEQUALITIES

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Abstract

Few different versions of Grüss type inequalities are given, using the Euler identity, Fink identity and Anastassiou identity.

1. Introduction

The following Grüss inequality (proved in 1935) is well known [7]:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right|$$
$$\leq\frac{1}{4}(M-m)(N-n).$$

It holds for every two integrable functions $f, g: [a,b] \to \mathbf{R}$ satisfying the condition

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N \quad \forall \in [a,b]$$

where m, M, n, N are given real constants.

In the recent paper [8] B. G. Pachpatte proved the following two Theorems

Theorem 1. Let $n \in N$, $f, g: [a,b] \to R$ be functions such that $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$ and $f^{(n)}, g^{(n)} \in L_\infty[a,b]$.

Then for $x \in [a,b]$ the following inequality holds

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big) \\ &+ \frac{1}{2(b-a)^2}\int_a^b \Big[\Big(\sum_{k=1}^{n-1} F_k(x)\Big)g(x) + \Big(\sum_{k=1}^{n-1} G_k(x)\Big)f(x)\Big]dx\Big| \qquad (1.1) \\ &\le \frac{1}{2(b-a)^2}\int_a^b \big(|g(x)|\,||f^{(n)}(t)||_\infty + |f(x)|\,||g^{(n)}(t)||_\infty\big)A_n(x)dx \end{aligned}$$

where

$$F_k(x) = \Big[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\Big]f^{(k)}(x), \qquad (1.2)$$

$$G_k(x) = \Big[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\Big]g^{(k)}(x). \qquad (1.3)$$

$$A_n(x) = \int_a^b |K_n(x,t)|dt$$

in which $\mathbf{K}_n\colon [a,b]^2 \to \mathbf{R}$ is given by

$$K_n(x,t) = \begin{cases} \dfrac{(t-a)^n}{n!} & \text{if} \quad t \in [a,x] \\ \dfrac{(t-b)^n}{n!} & \text{if} \quad t \in \langle x,b] \end{cases} \qquad (1.4)$$

Theorem 2. Let $n \in N$, $f,g\colon [a,b] \to R$ be function such that $f^{(n-1)}$, $g^{(n-1)}$ are absolutely continuous on $[a,b]$ and $f^{(n)}$, $g^{(n)} \in L_\infty[a,b]$. Then for $x \in [a,b]$ the following inequality holds

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - n\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big) \\ &+ \frac{1}{2(b-a)}\int_a^b \Big[\Big(\sum_{k=1}^{n-1} \tilde{F}_k(x)\Big)g(x) + \Big(\sum_{k=1}^{n-1} \tilde{G}_k(x)\Big)f(x)\Big]dx\Big| \qquad (1.5) \\ &\le \frac{1}{2(n-1)!(b-a)^2}\int_a^b (|g(x)|\,||f^{(n)}(t)||_\infty + |f(x)|\,||g^{(n)}(t)||_\infty)B_n(x)dx \end{aligned}$$

where

$$\tilde{F}_k(x) = \frac{(n-k)}{k!}\cdot\frac{(x-a)^k f^{(k-1)}(a) - (x-b)^k f^{(k-1)}(b)}{b-a} \qquad (1.6)$$

$$\tilde{G}_k(x) = \frac{(n-k)}{k!} \cdot \frac{(x-a)^k g^{(k-1)}(a) - (x-b)^k g^{(k-1)}(b)}{b-a} \tag{1.7}$$

$$B_n(x) = \int_a^b |(x-t)^{n-1} k(t,x)| dt$$

$$k(t,x) = \begin{cases} t-a, & \text{if} \quad a \le t \le x \le b \\ t-b, & \text{if} \quad a \le x \le t \le b \end{cases} \tag{1.8}$$

The main purpose of this paper is to generalize these two Grüss type inequalities and also to obtain a new one using Euler identity.

2. Some Further generalizations of Grüss inequality

Theorem 3. Let (p,q) be a pair of conjugate exponents i.e. $1 \le p$, $q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $f, g\colon [a,b] \to R$ functions such that $f^{(n-1)}, g^{(m-1)}$ are absolutely continuous on $[a,b]$ and $f^{(n)}, g^{(m)} \in L_p[a,b], n, m \in \mathbf{N}$. Then for $x \in [a,b]$ the following inequality holds

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big) \\ &+ \frac{1}{2(b-a)^2}\int_a^b \Big[\Big(\sum_{k=1}^{n-1} F_k(x)\Big)g(x) + \Big(\sum_{k=1}^{m-1} G_k(x)\Big)f(x)\Big]dx\Big| \qquad (2.1) \\ &\le \frac{1}{2(b-a)^2}\int_a^b (|g(x)|\ ||K_n(x,t)||_q ||f^{(n)}(t)||_p + |f(x)|\ ||K_m(x,t)||_q ||g^{(m)}(t)||_p) dx \end{aligned}$$

where $F_k(x), G_k(x)$ and $K_n(t,x)$ are given by (1.2), (1.3) and (1.4).

Proof. The following integral identity is valid (see 3)

$$f(x) = \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{b-a}\sum_{k=1}^{n-1} F_k(x) - \frac{(-1)^n}{b-a}\int_a^b K_n(x,t) f^{(n)}(t)dt$$

and similarly

$$g(x) = \frac{1}{b-a}\int_a^b g(x)dx - \frac{1}{b-a}\sum_{k=1}^{m-1} G_k(x) - \frac{(-1)^m}{b-a}\int_a^b K_m(x,t) g^{(m)}(t)dt$$

Multiplying the first equality by $g(x)$ and the second by $f(x)$, summing the resulting identities and integrating with respect to x on $[a,b]$, then dividing by $2(b-a)$, yields to

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx = \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$
$$-\frac{1}{2(b-a)^2}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}F_k(x)\Big)g(x)+\Big(\sum_{k=1}^{m-1}G_k(x)\Big)f(x)\Big]dx$$
$$-\frac{1}{2(b-a)^2}\int_a^b\Big((-1)^n g(x)\Big(\int_a^b K_n(x,t)f^{(n)}(t)dt\Big)$$
$$+(-1)^m f(x)\Big(\int_a^b K_m(x,t)g^{(m)}(t)dt\Big)\Big)dx\,.$$

From the properties of modulo we have

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$
$$+\frac{1}{2(b-a)^2}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}F_k(x)\Big)g(x)+\Big(\sum_{k=1}^{m-1}G_k(x)\Big)f(x)\Big]dx\Big|$$
$$\le\frac{1}{2(b-a)^2}\int_a^b\Big(|g(x)|\int_a^b|K_n(x,t)f^{(n)}(t)|dt$$
$$+|f(x)|\int_a^b|K_m(x,t)g^{(m)}(t)|dt\Big)dx\,.$$

After applying Hölder inequality

$$\int_a^b|K_n(x,t)f^{(n)}(t)|dt\le||K_n(x,t)||_q||f^{(n)}(t)||_p$$

$$\int_a^b|K_m(x,t)g^{(m)}(t)|dt\le||K_m(x,t)||_q||g^{(m)}(t)||_p$$

we obtain the inequality (2.1) □

Remark 1. In the special case for $m=n$ and $p=\infty$, $q=1$ the inequality from the last theorem reduces to inequality (1.1) obtained by B. G. Pachpatte.

Corollary 1. Suppose all the assumptions from the Theorem 3 holds. Then for $q>1$we have

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$

$$+\frac{1}{2(b-a)^2}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}F_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}G_k(x)\Big)f(x)\Big]dx\Big|$$

$$\leq\frac{(b-a)^{n+\frac{1}{q}-2}}{2n!(nq+1)^{\frac{1}{q}}}\int_a^b\Big(|g(x)|\;||f^{(n)}(t)||_p+|f(x)|\;||g^{(n)}(t)||_p\Big)dx$$

and for $q=1$

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$

$$+\frac{1}{2(b-a)^2}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}F_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}G_k(x)\Big)f(x)\Big]dx\Big|$$

$$\leq\frac{(b-a)^{n-2}}{2n!}\int_a^b\Big(|g(x)|\;||f^{(n)}(t)||_p+|f(x)|\;||g^{(n)}(t)||_p\Big)dx\,.$$

Proof. Since $(x-a)^{\alpha}+(b-x)^{\alpha}\geq(b-a)^{\alpha}$, for $\alpha>0$ and $x\in[a,b]$ (see [4], [6]), in case $q>1$ we have

$$\begin{aligned}||K_n(x,t)||_q&=\Big(\int_a^b|K_n(x,t)|^q dt\Big)^{1/q}\\&=\Big(\int_a^x\Big|\frac{(t-a)^n}{n!}\Big|^q dt+\int_x^b\Big|\frac{(t-b)^n}{n!}\Big|^q dt\Big)^{1/q}\\&=\frac{1}{n!}\Big(\int_a^x(t-a)^{nq}\,dt+\int_x^b(b-t)^{nq}dt\Big)^{1/q}\\&=\frac{1}{n!}\Big(\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\Big)^{1/q}\leq\frac{1}{n!}\Big(\frac{(b-a)^{nq+1}}{nq+1}\Big)^{1/q}\end{aligned}$$

and for $q=1$

$$||K_n(x,t)||_1=\sup_{t\in[a,b]}|K_n(x,t)|=\max\Big\{\frac{(x-a)^n}{n!},\frac{(b-x)^n}{n!}\Big\}\leq\frac{(b-a)^n}{n!}.$$

If we apply inequality (2.1) with n=m the inequality follows. □

Theorem 4. Let (p,q) be *a* pair of conjugate exponents, i.e. $1\leq p$, $q\leq\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, f,g: $[a,b]\to R$ functions such that $f^{(n-1)},g^{(n-1)}$ are absolutely continuous on $[a,b]$ and $f^{(n)},g^{(n)}\in L_p[a,b]$. Then the following inequality holds

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx-n\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$

$$+\frac{1}{2(b-a)}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}\tilde{F}_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}\tilde{G}_k(x)\Big)f(x)\Big]dx\Big| \tag{2.2}$$
$$\leq\frac{\|(x-t)^{n-1}k(t,x)\|_q}{2(n-1)!(b-a)^2}\int_a^b\Big(|g(x)|\,\|f^{(n)}(t)\|_p+|f(x)|\,\|g^{(n)}(t)\|_p\Big)dx$$

where $\tilde{F}_k(x), \tilde{G}_k(x)$ and $k(t,x)$ are given by (1.6), (1.7) and (1.8).

Proof. The Fink identity for function f states (see 5):

$$f(x)=\frac{n}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\tilde{F}_k(x)-\frac{1}{(n-1)!(b-a)}\int_a^b(x-t)^{n-1}k(t,x)f^{(n)}(t)dt$$

and similarly for g

$$g(x)=\frac{n}{b-a}\int_a^b g(x)dx-\sum_{k=1}^{n-1}\tilde{G}_k(x)-\frac{1}{(n-1)!(b-a)}\int_a^b(x-t)^{n-1}k(t,x)g^{(n)}(t)dt\,.$$

Multiplying the first equality by $g(x)$ and the second by $f(x)$, summing the resulting identities and integrating with respect to x on $[a,b]$, then dividing by $2(b-a)$, yields to

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx=n\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$
$$-\frac{1}{2(b-a)}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}\tilde{F}_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}\tilde{G}_k(x)\Big)f(x)\Big]dx$$
$$-\frac{1}{2(n-1)!(b-a)^2}\int_a^b\Big(g(x)\Big(\int_a^b(x-t)^{n-1}k(t,x)f^{(n)}(t)dt\Big)$$
$$+f(x)\Big(\int_a^b(x-t)^{n-1}k(t,x)g^{(n)}(t)dt\Big)\Big)dx\,.$$

From the properties of modulo we have

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx-n\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$
$$+\frac{1}{2(b-a)}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}\tilde{F}_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}\tilde{G}_k(x)\Big)f(x)\Big]dx\Big|$$
$$\leq\frac{1}{2(n-1)!(b-a)^2}\int_a^b\Big(|g(x)|\int_a^b|(x-t)^{n-1}k(t,x)f^{(n)}(t)|dt$$
$$+|f(x)|\int_a^b|(x-t)^{n-1}k(t,x)g^{(n)}(t)|dt\Big)dx\,.$$

After applying Hölder inequality

$$\int_a^b |(x-t)^{n-1}k(t,x)f^{(n)}(t)|dt \le ||(x-t)^{n-1}k(t,x)||_q||f^{(n)}(t)||_p$$

$$\int_a^b |(x-t)^{n-1}k(t,x)g^{(n)}(t)|dt \le ||(x-t)^{n-1}k(t,x)||_q||g^{(n)}(t)||_p$$

we obtain the inequality (2.2). □

Remark 2. In the special case for $p=\infty, q=1$the inequality from the last theorem reduces to inequality (1.5) obtained by B. G. Pachpatte.

The Beta function is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Corollary 2. Suppose all the assumptions from the Theorem 4 holds. Then for $q>1$ we have

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - n\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$
$$+\frac{1}{2(b-a)}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}\tilde{F}_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}\tilde{G}_k(x)\Big)f(x)\Big]dx\Big|$$
$$\le\frac{(b-a)^{n+\frac{1}{q}-2}B(q+1,q(n-1)+1)^{\frac{1}{q}}}{2(n-1)!}\int_a^b\Big(|g(x)|\,||f^{(n)}(t)||_p+|f(x)|\,||g^{(n)}(t)||_p\Big)dx$$

and for $q=1$

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - n\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)$$
$$+\frac{1}{2(b-a)}\int_a^b\Big[\Big(\sum_{k=1}^{n-1}\tilde{F}_k(x)\Big)g(x)+\Big(\sum_{k=1}^{n-1}\tilde{G}_k(x)\Big)f(x)\Big]dx\Big|$$
$$\le (b-a)^{n-2}\frac{(n-1)^{n-1}}{2(n-1)!n^n}\int_a^b\Big(|g(x)|\,||f^{(n)}(t)||_p+|f(x)|\,||g^{(n)}(t)||_p\Big)dx.$$

Proof. For $q>1$ we have

$$||(x-t)^{n-1}k(t,x)||_q=\Big(\int_a^b|(x-t)^{n-1}k(t,x)|^q dt\Big)^{1/q}$$
$$=\Big(\int_a^x|(x-t)^{n-1}(t-a)|^q dt+\int_x^b|(x-t)^{n-1}k(t-b)|^q dt\Big)^{1/q}.$$

Using substitution $t-a=u(x-a)$ the first integral is equal to

$$\int_a^x (x-t)^{q(n-1)}(t-a)^q dt = (x-a)^{nq+1}\int_0^1 u^q(1-u)^{q(n-1)}du$$
$$= (x-a)^{nq+1}B\big(q+1, q(n-1)+1\big).$$

Similarly, using substitution $b-t=u(b-x)$ the second integral is equal to

$$\int_x^b (t-x)^{q(n-1)}(b-t)^q dt = (b-x)^{nq+1}\int_0^1 u^q(1-u)^{q(n-1)}du$$
$$= (b-x)^{nq+1}B\big(q+1, q(n-1)+1\big).$$

So, we have

$$||(x-t)^{n-1}k(t,x)||_q \leq \Big(\big((x-a)^{nq+1}+(b-x)^{nq+1}\big)B(q+1,q(n-1)+1)\Big)^{1/q}$$
$$\leq \big((b-a)^{nq+1}B(q+1,q(n-1)+1)\big)^{1/q}$$
$$= (b-a)^{n+\frac{1}{q}}B\big(q+1,q(n-1)+1\big)^{1/q}.$$

For $q=1$, by simple calculation we get

$$||(x-t)^{n-1}k(t,x)||_1 = \sup_{t\in[a,b]} |(x-t)^{n-1}k(t,x)|$$
$$= \max\left\{(x-a)^n\frac{(n-1)^{n-1}}{n^n}, (b-x)^n\frac{(n-1)^{n-1}}{n^n}\right\} \leq (b-a)^n\frac{(n-1)^{n-1}}{n^n}.$$

If we apply inequality (2.2) the inequality follows. □

For every function $f\colon [a,b]\to R$ such that $f^{(n-1)}$ is a absolutely continuous function on $[a,b]$ for some $n\geq 1$ and for every $x\in[a,b]$, the following formula (Euler identity) is valid (see [4])

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_n(f;x) + P_n(f;x) \tag{2.3}$$

where

$$T_m(f;x) = \sum_{k=1}^{m}\frac{(b-a)^{k-1}}{k!}B_k\Big(\frac{x-a}{b-a}\Big)[f^{(k-1)}(b)-f^{(k-1)}(a)]$$

with convention $T_0(f;x)=0$, and

$$P_n(f;x) = -\frac{(b-a)^{n-1}}{n!}\int_a^b \Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]f^{(n)}(t)dt.$$

Here $B_k(x), k \geq 0$, are the Bernoulli polynomials, and $B_k^*(x), k \geq 0$, are periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leq x < 1, \quad B_k^*(x+1) = B_k^*(x), \quad x \in R.$$

Theorem 5. Let (p, q) be a pair of conjugate exponents, i.e. $1 \leq p$, $q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $f, g\colon [a, b] \to R$ functions such that $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a, b]$ and $f^{(n)}, g^{(n)} \in L_p[a, b]$. Then for $x \in [a, b]$ the following inequality holds

$$\begin{aligned}
&\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big) \\
&- \frac{1}{2(b-a)}\int_a^b [T_n(f;x)g(x) + T_n(g;x)f(x)]dx\Big| \qquad (2.4)\\
&\leq \frac{(b-a)^{n-2}}{2n!}\Big\|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]\Big\|_q \int_a^b \Big(|g(x)|\,\|f^{(n)}(t)\|_p + |f(x)|\,\|g^{(n)}(t)\|_p\Big)dx.
\end{aligned}$$

Proof. Using (2.3) we have

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_n(f;x) + P_n(f;x),$$

$$g(x) = \frac{1}{b-a}\int_a^b g(t)dt + T_n(g;x) + P_n(g;x),$$

Multiplying the first equality by $g(x)$ and the second by $f(x)$, summing the resulting identities and integrating whit respect to x on $[a, b]$, then dividing by $2(b-a)$, yields to

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)dx = \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big) \\
&+ \frac{1}{2(b-a)}\int_a^b [T_n(f;x)g(x) + T_n(g;x)f(x)]dx \\
&+ \frac{(b-a)^{n-2}}{2n!}\int_a^b \Big(g(x)\Big(\int_a^b \Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]f^{(n)}(t)dt\Big) \\
&+ f(x)\Big(\int_a^b \Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]g^{(n)}(t)dt\Big)\Big)dx.
\end{aligned}$$

From the properties of modulo we have

$$\begin{aligned}&\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)\\&-\frac{1}{2(b-a)}\int_a^b[T_n(f;x)g(x)+T_n(g;x)f(x)]dx\Big|\\&\le\frac{(b-a)^{n-2}}{2n!}\int_a^b\Big(|g(x)|\int_a^b\Big|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]f^{(n)}(t)\Big|dt\\&+|f(x)|\int_a^b\Big|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]g^{(n)}(t)\Big|dt\Big)dx\,.\end{aligned}$$

After applying Hölder inequality

$$\int_a^b\Big|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]f^{(n)}(t)\Big|dt\le\Big\|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]\Big\|_q||f^{(n)}(t)||_p$$

$$\int_a^b\Big|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]g^{(n)}(t)\Big|dt\le\Big\|\Big[B_n^*\Big(\frac{x-t}{b-a}\Big)\Big]\Big\|_q||g^{(n)}(t)||_p$$

we obtain inequality (2.4). □

Corollary 3. Suppose all the assumptions from the Theorem 5 holds. Then for $q>1$ we have

$$\begin{aligned}&\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)\\&-\frac{1}{2(b-a)}\int_a^b[T_n(f;x)g(x)+T_n(g;x)f(x)]dx\Big|\\&\le\frac{(b-a)^{n-2+\frac{1}{q}}}{2n!}\Big(\int_0^1|B_n(s)|^q ds\Big)^{\frac{1}{q}}\int_a^b(|g(x)|\,||f^{(n)}(t)||_p+|f(x)|\,||g^{(n)}(t)||_p)dx\end{aligned}$$

and for $q=1$

$$\begin{aligned}&\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big)\\&-\frac{1}{(2b-a)}\int_a^b[T_n(f;x)g(x)+T_n(g;x)f(x)]dx\Big|\\&\le\frac{(b-a)^{n-2+\frac{1}{q}}}{2n!}\Big(\max_{t\in[0,1]}|B_n(t)|\Big)\int_a^b\Big(|g(x)|\,||f^{(n)}(t)||_p+|f(x)|\,||g^{(n)}(t)||_p\Big)dx\,.\end{aligned}$$

Proof. For $q > 1$ we have

$$\left\| \left[B_n^* \left(\frac{x-t}{b-a} \right) \right] \right\|_q dt = \left(\int_a^b \left| B_n^* \left(\frac{x-t}{b-a} \right) \right|^q dt \right)^{1/q}.$$

Since B_n^* is a periodic function with period 1 and

$$\int_0^1 |B_n^*(y+s)| ds = \int_0^1 |B_n^*(s)| ds = \int_0^1 |B_n(s)| ds$$

for every $y \in R$, we have

$$\left(\int_a^b \left| B_n^* \left(\frac{x-t}{b-a} \right) \right|^q dt \right)^{1/q} = \left((b-a) \int_0^1 |B_n^*(y+s)|^q ds \right)^{1/q}$$
$$= \left((b-a) \int_0^1 |B_n(s)|^q ds \right)^{1/q}.$$

For $q = 1$, from the properties of Bernoulli polynomials we have

$$\left\| \left[B_n^* \left(\frac{x-t}{b-a} \right) \right] \right\|_1 = \sup_{t \in [a,b]} \left| B_n^* \left(\frac{x-t}{b-a} \right) \right| = \max_{t \in [0,1]} |B_n(t)|$$

and if we apply inequality (2.4) the inequality follows . □

References

[1] M. Abramowitz and I. A. Stegun (Eds): *Handbook of mathematical functions with formulae, graphs and mathematical tables.* National Bureau of Standards, Applied Math. Series 55, 4$^{\text{th}}$ printing, Washington 1965.

[2] A. Aglić Aljinović, Lj. Marangunić and J. Pečarić: *On Landau type inequalities via extension of Montgomery identity, Eueler and Fink identities,* Nonlinear functional analysis and applications (to appear).

[3] P. Cerone, S. S. Dragomir and J. Roumeliots: *Some Ostrowski type inequality for n-times differentiable mappings and applications,* Demonstratio Mathematica, 32 (2) (1999), 679-712.

[4] Lj. Dedić, M. Matić and J. Pečarić: *On generalizations of Ostrowski inequality via some Euler - type identities.* Math. Inequal. & Appl., 3 (3) (2000), 337-353.

[5] A. M. Fink: *Bounds of the derivation of a function from its averages,* Czechoslovak Math. J. 42 (117) (1992), 289-310.

[6] Lj. Marangunić and J. Pečarić: *On Landau type inequalities for functions with Hölder continuous derivatives,* JIPAM (to appear).

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink: *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[8] B. G. Pachpatte: *On Grüss type integral inequalities,* JIPAM, Volume 3, Issue 1, Article 11, (2002).

НЕКОИ ИНТЕГРАЛНИ НЕРАВЕНСТВА ОД ТИПОТ НА GRÜSS

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Р е з и м е

Во оваа работа се дадени докази на две теореми и тоа Теорема 3 и Теорема 4, кои ги генерализираат две интегрални неравенства од типот на Grüss. Исто така дадено е ново неравенство од типот на Grüss, тоа е Теоремата 5, кое е добиено со користење на идентитет на Eueler.

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