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FREE GROUPOIDS WITH $xy^2 = xy$

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Abstract

The main results of the paper are Theorems 1, 2, 3. Theorem 1 gives a canonical description of free objects in the variety $\mathcal{U}_r$ of groupoids which satisfy the identity $xy^2 = xy$. In Theorem 2 the class of $\mathcal{U}_r$–free groupoids is characterized within the class of $\mathcal{U}_r$–injective groupoids, which is larger than the class of $\mathcal{U}_r$–free groupoidsa Final , in Theorem 3, it is shown that the class of $\mathcal{U}_r$–free groupoids is hereditary, and that a $\mathcal{U}_r$–free groupoid with rank 2 contains subgroupoids with infinite rank.

0. Introduction

Throughout the paper we denote by $\boldsymbol{F} = (F, \cdot)$ a free groupoid (in the class of all groupoids) with a given basis B. It is well–known (for example [1; I.1]) that the following two properties characterize $\boldsymbol{F}$:

(a) *$\boldsymbol{F}$ is injective, i.e. the mapping* $\cdot : (a, b) \mapsto ab$ *is an injection from* F^2 *into* F.

(b) *The set B of primes in $\boldsymbol{F}$ generates $\boldsymbol{F}$.* (If $\boldsymbol{G} = (G, \cdot)$ is a groupoid, and $a \in G \setminus GG$, then we say that a is a *prime* in $\boldsymbol{G}$.)

As usual, if $\boldsymbol{G} = (G, \cdot)$ is a groupoid, and n is a positive integer, then the transformation $x \mapsto x^n$ is defined as follows:

$$x^1 = x\,, \qquad x^{k+1} = x^k x\,. \tag{0.1}$$

An element $a \in G$ is called a *proper power* in $\boldsymbol{G}$ iff there exist a $b \in \boldsymbol{G}$ and $n \in \mathbb{N}$, $n \geq 2$ ($\mathbb{N}$ is the set of positive integers), such that $a = b^n$. Then we say that b is a *base*, and n is an *exponent* of a in $\boldsymbol{G}$.

It can be easily shown by (a) and (0.1) that, if u is a proper power in $\boldsymbol{F}$, the base $t = \underline{u}$ and exponent $n = \mathrm{ex}(u)$ are unique. If $u \in F$ is not a proper power in $\boldsymbol{F}$, then we say that u is the *base* of u in $\boldsymbol{F}$, and write $\underline{u} = u$; in this case, 1 is the *exponent* of u in $\boldsymbol{F}$.

Notions as subgroupoids, homomorphisms, variety of groupoids, ... have usual meanings ([2]).

Now we can state the main results of the paper.

THEOREM 1. *Let* $\boldsymbol{R} = (R, *)$ *be defined as follows:*

$$B \subset R \subset F \;\&\; (\forall u,\, v \in F)\{uv \in R \Leftrightarrow u, v \in R \;\&\; \underline{v} = v\}\,, \tag{0.2}$$

$$(\forall u,\, v \in R)\, u * v = u\underline{v}\,. \tag{0.3}$$

Then $\boldsymbol{R}$ *is a free groupoid in* $\mathcal{U}_r$ *with the (unique) basis* B.

(We say that $\boldsymbol{R}$ is a *canonical* $\mathcal{U}_r$*-groupoid.*)

In order to state Th. 2, we will define the notion of $\mathcal{U}_r$-injectivity. Namely, we say that a groupoid $\boldsymbol{H} = (H, \cdot) \in \mathcal{U}_r$ is $\mathcal{U}_r$*-injective* iff it satisfies the following conditions:

1) $(\forall a \in H,\; n \in \mathbb{N})\; a \neq a^{n+1}$.

2) For each $a \in HH$ there is a unique pair $(b, c) \in H^2$ such that $a = bc$ and:

2.1) $(\forall d \in H,\; n \in \mathbb{N})\;\; c \neq d^{n+1}$.

2.2) $(\forall b',\, c' \in H)\; [a = b'c' \Rightarrow b' = b \;\&\; (c' = c^m$, for some $m \geq 1)]$.

In this case we say that b is *the left* and c is *the right divisor of* a (or shortly: (b, c) is *the pair of divisors of* a) and we write $b \mid a$, $c \mid a$. A sequence $a_1, a_2, \ldots$ of elements of H is called a *divisor chain in* $\boldsymbol{H}$ iff $a_{i+1} \mid a_i$ whenever a_{i+1} is a member of the sequence.

In Section **2** we give a complete description of the class of $\mathcal{U}_r$-injective groupoids, and show that it is larger than the class of $\mathcal{U}_r$-free [1)] groupoids. The following property is a description of $\mathcal{U}_r$-free groupoids within the class of $\mathcal{U}_r$-injective groupoids.

THEOREM 2. *If* $\boldsymbol{H} = (H, \cdot)$ *is a* $\mathcal{U}_r$*-injective groupoid, then the following conditions are equivalent:*

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1) We will often say "$\mathcal{U}_r$-free groupoid" instead of "free groupoid in $\mathcal{U}_r$".

(i) $\boldsymbol{H}$ is *$\mathcal{U}_r$–free.*

(ii) *There is a mapping* $|\ |: a \mapsto |a|$ *from* H *into the set* $\mathbb{N}$ *of positive integers such that:* $b \mid a \Rightarrow |b| < |a|$.

(iii) *Every divisor chain in* $\boldsymbol{H}$ *is finite.*

(iv) *The set* B *of primes in* $\boldsymbol{H}$ *generates* $\boldsymbol{H}$.
Then B *is the basis of* $\boldsymbol{H}$.

THEOREM 3. (1) *The class of $\mathcal{U}_r$–injective groupoids and the class of $\mathcal{U}_r$–free groupoids are hereditary.*

(2) *If* $\boldsymbol{H}$ *is a $\mathcal{U}_r$–free groupoid with rank one, then each subgroupoid of* $\boldsymbol{H}$ *is infinite and isomorphic to* $\boldsymbol{H}$.

(3) *If* $\boldsymbol{H}$ *is a $\mathcal{U}_r$–free groupoid with rank two, then there exists subgroupoids of* $\boldsymbol{H}$ *with infinite rank.*

Theorem i ($i = 1, 2, 3$) (beside other auxiliary results) will be proved in Section i.

SOME REMARKS

1. The axion $xy^2 = xy$ of $\mathcal{U}_r$ suggests to consider the rewriting system (RS) on F induced by the elementary transformation $uv^2 \to uv$. Clearly, this system is terminating (T) but it is not Church–Rosser (CR) one (see [5; 2.9, 3.5]). For example, we have: $a \cdot a^2a^2 \to a \cdot a^2a$ and $a \cdot a^2a^2 = a(a^2)^2 \to aa^2 \to a^2$. But, if we allow each transformation of the form $uv^k \to uv$, where $k \geq 2$, then we would obtain the corresponding RS which is a convenient TCR. We note that RS–s induced by $x^2y^2 \to (xy)^2$ (i.e. $x^n \to x$) are convenient TCR for the variety $\mathcal{V}_2$ ($\mathcal{V}$) defined by $x^2y^2 = (xy)^2$ ($x^n = x,\ n \geq 2$).

2. In [3], [4] corresponding Th. 1, Th. 2, Th. 3 for the varieties $\mathcal{V}_2$ and $\mathcal{V}$ are shown. The formulation of these theorems for $\mathcal{V}_2$ ([3]) and $\mathcal{V}$ ([4]) are almost the same as for $\mathcal{U}_r$, except Th. 3 for $\mathcal{V}_2$ (the class of $\mathcal{V}_2$–free groupoids is not hereditary).

3. Denote by $\mathcal{U}_l$ the variety of groupoids with the identity $x^2y = xy$. Clearly:

$$\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_l \ \Leftrightarrow\ \boldsymbol{G}^{\mathrm{op}} = (G, \circ) \in \mathcal{U}_r\,,$$

where $x \circ y = yx$. Therefore, each $\mathcal{U}_r$–property can be translated into corresponding $\mathcal{U}_l$–property.

1. Canonical $\mathcal{U}_r$–groupoids

A proof of Th. 1 will be given below.

First, let $u \mapsto |u|$ be the homomorphism from F into the groupoid $(\mathbb{N}, +)$ which is an extension of $B \to \{1\}$. Then:

$$|b| = 1 \qquad |uv| = |u| + |v|, \tag{1.1}$$

for any $b \in B$ and $u, v \in F$. (We say that $|u|$ is the *norm* of $u \in F$.) By induction on norm, the following relation can be easily shown:

$$(\forall u, v \in F, \; p, q \in \mathbb{N}) \; [u^{p+1} = v^{q+1} \;\Rightarrow\; u = v, \; p = q], \tag{1.2}$$

and this implies that $u \mapsto \underline{u}$, where $\underline{u}$ is the base of u, is a well defined transformation of F, such that

$$(\forall v \in F) \underline{v \, \underline{v}} = \underline{v}. \tag{1.3}$$

Moreover, (0.2), (0.3) and (1.1) imply:

$$v \in R \;\Rightarrow\; \underline{v} \in R, \tag{1.4}$$

$$n \geq 2 \;\Rightarrow\; (v^n \in R \;\Leftrightarrow\; v = \underline{v} \in R), \tag{1.5}$$

$$\begin{gathered} (\forall u, v \in R) \; |u| + 1 \leq |u| + |\underline{v}| \leq |u * v| \leq |u| + |v|, \\ |u * v| = |u| + |v| \;\Leftrightarrow\; v = \underline{v}. \end{gathered} \tag{1.6}$$

As a corollary from (0.2), (0.3) and (1.4) we obtain:

1.1. $*\colon R^2 \to R$ *is a well defined mapping, i.e.* $\boldsymbol{R}$ *is a groupoid.* □

Moreover, from (0.3) and (1.3) we obtain:

$$(\forall u, v \in R) \; u * (v * v) = u * (v \, \underline{v}) = u \, \underline{v \, \underline{v}} = u \, \underline{v} = u * v.$$

Therefore:

1.2. $\boldsymbol{R} \in \mathcal{U}_r$. □

It is also clear that:

1.3. *B is the least generating subset of* $\boldsymbol{R}$. □

In completing the proof of Th. 1 we will use the next two properties of $\mathcal{U}_r$.

1.4. *The following identities hold in* $\mathcal{U}_r$:

$$xy^n = xy, \quad x^m x^n = x^{m+1}, \quad (x^m)^n = x^{m+n-1}, \qquad \textit{for any } n,\, m \in \mathbb{N}.$$

Proof. Assuming $xy^n = xy$, we obtain:

$$xy^{n+1} = x \cdot y^n y = x \cdot y^n y^n = x(y^n)^2 = xy^n = xy.$$

The other two identities are trivial corollaries of the first one. □

1.5. *If* $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_r$, *and* φ *is a homomorphism from* $\boldsymbol{F}$ *into* $\boldsymbol{G}$, *then*:

$$(\forall\, u,\, v \in F)\ \varphi(uv) = \varphi(u\,\underline{v}).$$

Proof. Let $u,\, v \in F$ be such that $\mathrm{ex}(v) = n$, i.e. $v = (\underline{v})^n$. Then:

$$\varphi(uv) = \varphi(u)\,\varphi(v) = \varphi(u)\,\varphi\big((\underline{v})^n\big) = \varphi(u)\,\varphi\big((\underline{v})\big)^n = \varphi(u)\,\varphi(\underline{v}) = \varphi(u\,\underline{v}).\ \square$$

From **1.5** we obtain the following corollary:

1.6. *Let* $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_r$, $\lambda\colon B \to G$ *and* φ *be the homomorphism from* $\boldsymbol{F}$ *into* $\boldsymbol{G}$ *which extends* λ. *Then the restriction* ψ *of* φ *on* R *is a homomorphism from* $\boldsymbol{R}$ *into* $\boldsymbol{G}$, *which extends* λ. □

Finally, Th. 1 is a corollary of **1.2, 1.3** and **1.6**.

The following properties will be used in the next sections.

1.7. $\boldsymbol{R}$ *is* $\mathcal{U}_r$*-injective and* $(v,\, w)$ *is the pair of divisors of* $u \in R * R$ *iff*

$$|u| = |v| + |w|.$$

Proof. If $u \in R$, $k \in \mathbb{N}$, then we denote by u_*^k the k-th power of u in $\boldsymbol{R}$, i.e.

$$u_*^1 = u, \qquad u_*^{k+1} = u_*^k * u. \tag{1.7}$$

By (1.6), we have: $n \geq 2 \ \Rightarrow\ |u_*^n| > |u|$, and this implies that the condition 1) from Section **0** holds. If $u \in R{*}R$, then $u = v{*}w = vw$, where $v,\, w \in R$ and $\underline{w} = w$. Then $u = v' * w'$ iff $v' = v$ and $\underline{w'} = \underline{w} = w$. This implies that the condition 2) of Section **0** is satisfied, as well. □

The following two properties are also clear.

1.8. *If the operation* $\bullet$ *is defined in* $\mathbb{N}$ *as follows*:

$$(\forall\, m,\, n \in \mathbb{N}) \quad m \bullet n = m + 1, \tag{1.8}$$

then $(\mathbb{N}, \bullet)$ *is a* $\mathcal{U}_r$*-free grupoid with the basis* $\{1\}$. *The family of subgroupoids of* $(\mathbb{N}, \bullet)$ *is infinite, and each of them is isomorphic to* $(\mathbb{N}, \bullet)$. □

1.9. *If* $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_r$, *and* $a \in G$, *then the subgroupoid* $\boldsymbol{Q} = < a >$ *of* $\boldsymbol{G}$ *generated by* a *is determined as follows*:

$$Q = \{a^n \mid n \in \mathbb{N}\}, \quad a^m a^n = a^{m+1}. \tag{1.9}$$

And, Q *is* $\mathcal{U}_r$*-free with basis* $\{a\}$ *iff*:

$$(\forall\, m, n \in \mathbb{N}) \quad (a^m = a^n \;\Rightarrow\; m = n). \; \square \tag{1.10}$$

(As usual we say that $<a>$ is the *cyclic subgroupoid* of $\boldsymbol{G}$, generated by a.)

2. $\mathcal{U}_r$–injective groupoids

Below we assume that $\boldsymbol{H} = (H, \cdot)$ is a $\mathcal{U}_r$–injective groupoid, and: $a, b, c, d \in H$, $m, n, k \in \mathbb{N}$.

Using the implication: $xy = x'y' \;\Rightarrow\; x = x'$, and the definition of the class of $\mathcal{U}_r$–injective groupoids, the statements that follow can be easily shown.

$$a^n = b^n \;\Rightarrow\; a = b. \tag{2.1}$$

$$a^{m+1} = b^{m+n} \;\Rightarrow\; a = b^n. \tag{2.2}$$

$$a^m = a^n \;\Rightarrow\; m = n. \tag{2.3}$$

As a collorary of **1.9** and (2.3), we obtain:

2.1. *The subgroupoid* $< a >$ *of* $\boldsymbol{H}$, *generated by* $a \in \boldsymbol{H}$ *is* $\mathcal{U}_r$*-free with the basis* $\{a\}$. □

2.2. *For every* $a \in H$ *there is a unique pair* (b, n), *such that*

$$a = b^n \quad \text{and} \quad (b = c^m \;\Rightarrow\; m = 1).$$

(As in the groupoid $\boldsymbol{F}$ we say that b is *the base* and n *the exponent* of a, and use the following notations: $b = \underline{a}$, $n = \text{ex}(a)$.)

Proof. Assume that there exists a pair (b, n), such that $a = b^n$ and $n \geq 2$. Then, the right divisor c of a is the base of a. From $b^{n-1}b = b^{n-1}c$ it follows that there exists $m \in \mathbb{N}$ such that $b = c^m$, and therefore $a = (c^m)^n = c^{m+n-1}$, which implies that $\text{ex}(a) = m + n - 1$. □

The following two statements are clear.

$$(\underline{a}^n) = \underline{a}, \quad \mathrm{ex}(a^m) = m - 1 + \mathrm{ex}(a)\,. \tag{2.4}$$

$$a^m = b^n \;\Rightarrow\; \underline{a} = \underline{b}\,. \tag{2.5}$$

As a corollary from **2.1** and (2.5) we obtain:

2.3. *A cyclic subgroupoid* $<a>$ *of* $\boldsymbol{H}$ *is maximal iff* $\underline{a} = a$*; and, any two distinct maximal cyclic subgroupoids of* $\boldsymbol{H}$ *are disjoint.* □

2.4. *If* $\underline{a} = a,\ \underline{b} = b,\ a \neq b,\ n \in \mathbb{N}$ *and* $c = a^n b$, *then* $\underline{c} = c$.

Proof. Assume that $\underline{c} = d \neq c$. Then $c = d^{m+1}$, where $m + 1 = \mathrm{ex}(c) \geq 2$, and therefore $b = d,\ a^n = d^m$; by (2.5), $a^n = d^m$ implies $a = \underline{a} = \underline{d} = d = b$, a contradiction. □

2.5. *If the subset* $A \subseteq H$ *is defined by*

$$A = \{\underline{a} \mid a \in H\}\,, \tag{2.6}$$

then A *is either singleton or infinite.*

Proof. If A contains at least two distinct elements, then by **2.4**, A is infinite. □

2.6. *Let* ψ *be the mapping from* $(H \times \mathbb{N}) \times H$ *into* H *defined by*

$$\psi\big((a,\, n),\, b\big) = a^n b\,, \tag{2.7}$$

and

$$D = (A \times \mathbb{N}) \times A \setminus \big\{\big((a,\, n),\, a\big) \mid a \in A,\ n \in \mathbb{N}\big\}\,, \tag{2.8}$$

where A *is defined in* (2.6). *Then the restriction* φ *of* ψ *on* D *is injective and* $im\varphi \subseteq A$.

Proof. The inclusion $im\varphi \subseteq A$ follows from **2.4**. If $a, b, c, d \in A$, $m, n \in \mathbb{N}$ are such that $a \neq b,\ c \neq d,\ a^n b = c^m d$, then $b = d$, and $a^n = c^m$, and therefore: $a = c,\ m = n$. (Note that if A is a singleton set, then $D = \emptyset$.) □

The last result suggests the following *construction*.

Let A be a singleton or an infinite set, and let $M = A \times \mathbb{N}$, where the equality $(a,\, 1) = a$, for each $a \in A$ is assumed. Let $\varphi\colon \big((a,\, n),\, b\big) \mapsto \varphi\big((a,\, n),\, b\big)$ be an injection from the set (2.8) into A. Define an operation $\bullet$ on M as follows:

$$(a,\, m) \bullet (a,\, n) = (a,\, m + 1)\,, \tag{2.9}$$

$$a \neq b \;\Rightarrow\; (a,\, m) \bullet (b,\, n) = \varphi\big((a,\, m),\, b\big)\,. \tag{2.10}$$

Denote by $(A,\, \varphi)$ the groupoid $\boldsymbol{M} = (M,\, \bullet)$.

The following *characterization of $\mathcal{U}_r$–injective groupoids* can be easlily shown.

2.7. *(A, φ) is a $\mathcal{U}_r$–injective groupoid, such that*

$$A = \{\underline{(a, n)} \mid a \in A, \quad n \in \mathbb{N}\}, \tag{2.6'}$$

and $A \setminus im\varphi$ is the set of primes in (A, φ).

Conversely, let $\boldsymbol{H}$ be a $\mathcal{U}_r$–injective groupoid and A be defined by (2.6). *Then $\boldsymbol{H}$ is isomorphic to (A, φ), where φ is the restriction on D of the mapping ψ, defined by* (2.7). □

2.8. *The class of $\mathcal{U}_r$–free groupoids is a proper subclass of the class of $\mathcal{U}_r$–injective groupoids.*

Proof. By **1.7**, the class of $\mathcal{U}_r$–free groupoids is a subclass of the class of $\mathcal{U}_r$–injective groupoids. Let A be an infinite set. Then there exist groupoids (A, φ) such that $im\varphi = A$, and thus the set of primes in (A, φ) is empty. Therefore (A, φ) is not $\mathcal{U}_r$–free. □

2.9. *A groupoid (A, φ) is $\mathcal{U}_r$–free iff the set of primes generates (A, φ).*

Proof. If (A, φ) is $\mathcal{U}_r$–free, then the set of primes generates (A, φ) by Th. 1. Assume that $B = A \setminus im\varphi$ (the set of primes in (A, φ)) generates (A, φ). By **2.1**, if $B = \{b\}$ is a singleton set, (A, φ) is $\mathcal{U}_r$–free. It remains the case when B contains at least two distinct elements. Then, A is infinite.

Define a sequence of sets $\{B_k \mid k \geq 1\}$ as follows: $B = B_1$,

$$c \in B_{k+1} \quad \Leftrightarrow \quad c = \varphi\big((a, n), b\big), \tag{2.11}$$

where:

$$a \neq b, \quad n \in \mathbb{N}, \quad a \in B_i, \quad b \in B_j, \quad i, j \leq k, \quad k \in \{i, j\}. \tag{2.12}$$

The relations $B \cap im\varphi = \emptyset$, $im\varphi \subseteq A$, (2.11), (2.12), and the fact that φ is injective, imply:

$$B_{k+1} \cap (\cup\{B_i \mid 1 \leq i \leq k\}) = \emptyset, \tag{2.13}$$

and $\cup\{B_k \mid k \geq 1\} = A$, where the union is disjoint.

Let $\boldsymbol{G} \in \mathcal{U}_r$, and $\lambda: B \to G$. Define a set of mappings $\{\alpha_k: B_k \to G \mid k \geq 1\}$ as follows:

$$\alpha_1 = \lambda, \qquad \alpha_{k+1}(d) = \alpha_i(a)^n \alpha_j(b), \tag{2.14}$$

where $d = \varphi\big((a, n), b\big) \in D_{k+1}$, $a \in B_i$, $b \in B_j$, $n \in \mathbb{N}$.

There is a unique mapping $\alpha: A \to G$ such that, for each $k \in \mathbb{N}$, α_k is the restriction of α on B_k. Finally, the mapping $\overline{\lambda}: (A, \varphi) \to G$ defined by:

$$\overline{\lambda}((a, n)) = \alpha(a)^n$$

is a homomorphism which extends λ. □

Now we can complete the proof of Th. 2.

Assume that $\boldsymbol{H}$ is a $\mathcal{U}_r$–injective groupoid.

By **1.7**, (i) $\Rightarrow$ (ii). (Namely, if $\boldsymbol{H}$ is $\mathcal{U}_r$–free, then it is isomorphic to the coresponding canonical $\mathcal{U}_r$–groupoid.) Clearly, (ii) $\Rightarrow$ (iii).

Assume that $\boldsymbol{H}$ satifies (iii), i.e. every divisor chain in $\boldsymbol{H}$ is finite. From the $\mathcal{U}_r$–injectivity of $\boldsymbol{H}$, it follows that any element of $\boldsymbol{H}$ has at most two distinct divisors, and this, by an application of König Lemma (for example [6; 381] or [7; 4]) implies that the set of divisor chains in $\boldsymbol{H}$ with the same first member a is finite. Then the last members of such maximal divisor chains are primes in $\boldsymbol{H}$ and a belongs to the subgroupoid generated by them. Therefore, the set B of primes in $\boldsymbol{H}$ generates $\boldsymbol{H}$. Thus (iii) $\Rightarrow$ (iv). From **2.8** we also obtain that (iv) $\Rightarrow$ (i). This completes the proof of Th. 2. □

3. Subgroupoids of $\mathcal{U}_r$–free groupoids

The following statement is "a half" of the first part of Th. 3.

3.1. *The class of $\mathcal{U}_r$–injective groupoids is hereditary.*

Proof. Let $\boldsymbol{H}$ be a $\mathcal{U}_r$–injective groupoid and $\boldsymbol{Q}$ a subgroupoid of $\boldsymbol{H}$. We will show that $\boldsymbol{Q}$ is $\mathcal{U}_r$–injective . Clearly, the condition 1), in the definition of the class of $\mathcal{U}_r$–injective groupoids, is hereditary, and thus it remains to show that $\boldsymbol{Q}$ satisfies the condition 2).

Let $a \in QQ$. Then there exist $b', c' \in Q$ such that $a = b'c'$. If (b, c) is the pair of divisors of a in $\boldsymbol{H}$, then $b = b'$, and $c' = c^n$, for a (unique) $n \in \mathbb{N}$. Let k be the least positive integer such that $d = c^k \in Q$. Then $k \leq n$ and $c' = d^{n-k+1}$. This implies that $\boldsymbol{Q}$ satisfies the condition 2) as well. Namely, if $a \in QQ$, and (b, c) is the pair of divisors of a in $\boldsymbol{H}$, then (b, c^k) is the pair of divisors of a in $\boldsymbol{Q}$. □

In **3.2** and **3.3** we assume that $\boldsymbol{H}$ is a $\mathcal{U}_r$–injective groupoid, and $\boldsymbol{Q}$ a subgoupoid of $\boldsymbol{H}$.

3.2. *If $a \in Q$, and $\underline{a}_Q$ is the base of a in $\boldsymbol{Q}$, then there is a (unique) $k \in \mathbb{N}$ such that $\underline{a}_Q = (\underline{a})^k$, where $\underline{a}$ is the base of a in $\boldsymbol{H}$.* □

3.3. *If $a \in Q$ is such that $\underline{a}_Q = a = (\underline{a})^k$, where $k \geq 2$, then a is prime in $\boldsymbol{Q}$.*

Proof. Namely, the assumption that $a = (\underline{a})^k$ is not a prime in Q would imply that $(\underline{a})^{k-1} \in Q$. □

3.4. *If* $\boldsymbol{Q}$ *is a subgroupoid of a* $\mathcal{U}_r$*–free groupoid* $\boldsymbol{H}$*, then the set* P *of primes in* $\boldsymbol{Q}$ *generates* $\boldsymbol{Q}$.

Proof. By **1.7** $\boldsymbol{H}$ is $\mathcal{U}_r$–injective and there exists a mapping $a \mapsto |a|$ from H into $\mathbb{N}$ such that: if $a \in HH$ and (b, c) is the pair of divisors of a in $\boldsymbol{H}$, then $|a| = |b| + |c|$. By **3.1**, $\boldsymbol{Q}$ is $\mathcal{U}_r$–injective, and if $a \in QQ$ and (b, c) is the pair of divisors of a in $\boldsymbol{H}$, then there exists a (unique) $k \in \mathbb{N}$ such that (b, c^k) is the pair of divisors of a in $\boldsymbol{Q}$.

Let m be the least positive integer such that $Q \cap \{a \mid a \in H, |a| = m\} = S$ is non-empty. Then $S \subseteq P$ and thus $P \neq \emptyset$. Denote by T the subgroupoid of $\boldsymbol{Q}$ generated by P, and assume that

$$a \in Q \ \& \ |a| \leq n \ \Rightarrow \ a \in T .$$

Let $a \in Q$ and $|a| = n + 1$. We will show that $a \in T$. Clearly, $a \in P \Rightarrow a \in T$, and thus we can assume that $a \in QQ$. Let (b, c) be the pair of divisors of a in $\boldsymbol{H}$. By **1.7**, we have $|a| = |b| + |c|$ and thus $|b|, |c| \leq n$. By the proof of **3.1**, there is a (unique) $k \in \mathbb{N}$ such that (b, c^k) is the pair of divisors of a in $\boldsymbol{Q}$, and thus $b, c^k \in Q$, and $|b| \leq n$. If $k = 1$, then $c \in Q$ as well, and thus $a \in T$. Finally, if $k \geq 2$, then by **3.3**, $c^k \in P$, and therefore $a = bc^k \in T$. □

Now we can complete the proof of the first part of Th. 3.

3.5. *The class of* $\mathcal{U}_r$*–free groupoids is hereditary.*

Proof. Let $\boldsymbol{Q}$ be a subgroupoid of a $\mathcal{U}_r$–free groupoid $\boldsymbol{H}$. By **1.7**, **3.1**, **3.4** and Th. 2, $\boldsymbol{Q}$ is $\mathcal{U}_r$–free. □

The second part of Th. 3 follows from **1.8**, and the third one is a corollary from the following proposition.

3.6. *Let:* $\boldsymbol{H} = (H, \cdot)$ *be a* $\mathcal{U}_r$*–free groupoid,* $a, b \in H$ *be such that* $\underline{a} = a \neq b = \underline{b}$ *and* $C = \{C_k \mid k \geq 1\}$ *be defined by:*

$$c_1 = ab, \quad c_{k+1} = c_k b . \tag{3.2}$$

Then the subgroupoid $\boldsymbol{Q}$ *generated by* C *is* $\mathcal{U}_r$*–free with infinite rank.*

Proof. By **3.5**, $\boldsymbol{Q}$ is $\mathcal{U}_r$–free. By induction on $m + n$, one can show that: $c_m = c_n \Leftrightarrow m = n$, and thus C is infinite. Clearly $a \notin \boldsymbol{Q}$, $b \notin \boldsymbol{Q}$, and this implies that C coincides with the set of primes in $\boldsymbol{Q}$. (Namely, (a, b) is the pair of divisors of c_1 in $\boldsymbol{H}$, and $a \notin \boldsymbol{Q}$, $b \notin \boldsymbol{Q}$; this implies that c_1 is a prime in $\boldsymbol{Q}$; assuming that c_k is a prime in $\boldsymbol{Q}$, we obtain in the same way that c_{k+1} is also prime in $\boldsymbol{Q}$. □

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СЛОБОДНИ ГРУПОИДИ СО $xy^2 = xy$

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Р е з и м е

Главните резултати во работава се Теоремите 1, 2 и 3. Во теоремата 1 се дава каноничен опис на слободните објекти во многуобразието $\mathcal{U}_r$ од групоиди коишто го задоволуваат идентитетот $xy^2 = xy$. Во Теоремата 2 е окарактеризирана класата $\mathcal{U}_r$–слободни групоиди во рамките на класата $\mathcal{U}_r$–инјективни групоиди. На крајот, во Теоремата 3 е покажано дека секоја од споменатите класи е наследна и дека секој $\mathcal{U}_r$–слободен групоид со ранг 2 содржи подгрупоиди со бесконечен ранг.

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