Math. Maced.
Vol. 7 (2009)
21–28

EXTREMALLY DISCONNECTEDNESS AND SUBMAXIMALITY VIA $(1,2)^*$-OPEN SETS

M. LELLIS THIVAGAR[1] AND NIRMALA MARIAPPAN[2]

Abstract. The aim of this paper is to introduce $(1,2)^*$-extremally disconnectedness and $(1,2)^*$-submaximality in $(1,2)^*$-bitopological spaces and study their properties.

1. Introduction

Levine [3], Mashhour et al [6] and Njastad [7] have introduced the concepts of semi-open sets, preopen sets and α-open sets respectively. Levine [4] introduced generalised closed sets and studied their properties. Bhattacharya and Lahiri [2] introduced semi-generalised closed sets. Thivagar et al [8] have introduced the concepts of $(1,2)^*$-semi-open sets, $(1,2)^*$-generalised closed sets, $(1,2)^*$-semi-generalised closed sets in bitopological spaces. The aim of this paper is to introduce $(1,2)^*$-extremally disconnectedness and $(1,2)^*$-submaximality in $(1,2)^*$-bitopological spaces and study their properties.

2. Preliminaries

Throughout this paper (X, τ_1, τ_2) represents a bitopological space on which no separation axioms are assumed unless otherwise mentioned.

Definition 2.1. [8] *A subset S of a bitopological space (X, τ_1, τ_2) is said to be $\tau_{1,2}$-open if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$. A subset S of X is said to be $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open.*

Definition 2.2. [8] *Let S be a subset of X. Then*

(i) The $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-int(S), is defined by $\cup\{G/G \subset S$ and G is $\tau_{1,2}$-open $\}$.
(ii) The $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-cl(S), is defined by $\cap\{F/S \subset F$ and F is $\tau_{1,2}$-closed $\}$.

Remark 2.1.

(i) $\tau_{1,2}$-int(S) is $\tau_{1,2}$-open for each $S \subset X$ and $\tau_{1,2}$-cl(S) is $\tau_{1,2}$-closed for each $S \subset X$.

2010 *Mathematics Subject Classification.* 54E55, 54D10.
Key words and phrases. $(1,2)^*$-extremally disconnectedness and $(1,2)^*$-submaximality.

(ii) *A set* $S \subset X$ *is* $\tau_{1,2}$*-open iff* $S = \tau_{1,2}$*-int*(S) *and is* $\tau_{1,2}$*-closed iff* $S = \tau_{1,2}$*-cl*(S).

(iii) $\tau_{1,2}$*-int*$(S) = int_{\tau_1}(S) \cup int_{\tau_2}(S)$ *and* $\tau_{1,2}$*-cl*$(S) = cl_{\tau_1}(S) \cap cl_{\tau_2}(S)$ *for any* $S \subset X$

(iv) *For any family* $\{S_i/i \in I\}$ *of subsets of* X *we have*
 - (a) $\bigcup_i \tau_{1,2}\text{-}int(S_i) \subset \tau_{1,2}\text{-}int(\bigcup_i S_i)$
 - (b) $\bigcup_i \tau_{1,2}\text{-}cl(S_i) \subset \tau_{1,2}\text{-}cl(\bigcup_i S_i)$
 - (c) $\tau_{1,2}\text{-}int(\bigcap_i S_i) \subset \bigcap_i \tau_{1,2}\text{-}int(S_i)$
 - (d) $\tau_{1,2}\text{-}cl(\bigcap_i S_i) \subset \bigcap_i \tau_{1,2}\text{-}cl(S_i)$

(v) $\tau_{1,2}$*-open sets need not form a topology.*

We recall the following definitions which are useful in the sequel.

Definition 2.3. [8] *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is called*

(i) $(1,2)^*$*-semi-open if* $A \subseteq \tau_{1,2}\text{-}cl(\tau_{1,2}\text{-}int(A))$

(ii) $(1,2)^*$*-preopen if* $A \subseteq \tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(A))$

(iii) $(1,2)^*$*-*α*-open if* $A \subseteq \tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(\tau_{1,2}\text{-}int(A)))$

(iv) $(1,2)^*$*-semi-closed if* A^c *is* $(1,2)^*$*- semi-open.*

(v) $(1,2)^*$*-generalised closed (briefly* $(1,2)^*$*-g-closed) if* $\tau_{1,2}\text{-}cl(A) \subset U$ *whenever* $A \subset U$ *and* U *is* $\tau_{1,2}$*-open in* X.

(vi) $(1,2)^*$*-semi-generalised closed (briefly* $(1,2)^*$*-sg-closed) if* $(1,2)^*\text{-}scl(A) \subset U$ *whenever* $A \subset U$ *and* U *is* $(1,2)^*$*-semi-open in* X.

(vii) $(1,2)^*$*-sg-open if* A^c *is* $(1,2)^*$*-sg-closed.*

Definition 2.4. [8]

(i) *The* $(1,2)^*$*-semi-closure of a subset* A *of* X*, denoted by* $(1,2)^*\text{-}scl(A))$*, is defined to be the intersection of all* $(1,2)^*$*-semi-closed sets containing* A.

(ii) *The* $(1,2)^*$*-semi-interior of a subset* A *of* X*, denoted by* $(1,2)^*\text{-}sint(A)$*, is defined to be the union of all* $(1,2)^*$*-semi-open sets contained in* A.

Remark 2.2.

(i) *Since arbitrary union (resp. intersection) of* $(1,2)^*$*-semi-open (resp.*$(1,2)^*$*-semi-closed) sets is* $(1,2)^*$*-semi-open (resp.*$(1,2)^*$*-semi-closed),* $(1,2)^*\text{-}sint(A)$ *(resp.*$(1,2)^*\text{-}scl(A)$*) is* $(1,2)^*$*-semi-open (resp.*$(1,2)^*$*-semi-closed).*

(ii) *A subset* A *of* X *is* $(1,2)^*$*-semi-open (resp.* $(1,2)^*$*-semi-closed) if and only if* $(1,2)^*\text{-}sint(A)$ *(resp.* $(1,2)^*\text{-}scl(A)) = A$.

3. $(1,2)^*$-Extremally Disconnectedness

The following results in $(1,2)^*$-bitopological spaces will be useful for the charecterisation of $(1,2)^*$-extremally disconnected bitopological spaces.

Definition 3.1. *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is called*

(i) $(1,2)^*$*-nowhere dense if* $\tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(A)) = \phi$.

(ii) $(1,2)^*$*-dense if* $\tau_{1,2}\text{-}cl(A) = X$.

Theorem 3.1. *Every singleton set $\{x\}$ of a bitopological space (X,τ_1,τ_2) is either $(1,2)^*$-nowhere dense or $(1,2)^*$-preopen.*

Proof. Let $x \in X$. If $\{x\}$ is not $(1,2)^*$-nowhere dense then $G = \tau_{1,2}$-int($\tau_{1,2}$-cl($\{x\}$)) $\neq \phi$. Suppose x is not in G. Then G^c contains x. Since G^c is $\tau_{1,2}$-closed, $G^c \supseteq \tau_{1,2}$-cl($\{x\}$) $\supseteq G$ which implies $G = \phi$, a contradiction. Hence $\{x\} \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl($\{x\}$)) or $\{x\}$ is $(1,2)^*$-preopen.

Theorem 3.1 provides a decomposition $X = X_1 \cup X_2$ of (X,τ_1,τ_2) where $X_1 = \{x \in X : \{x\}$ is $(1,2)^*$-nowhere dense$\}$ and $X_2 = \{x \in X : \{x\}$ is $(1,2)^*$-preopen$\}$. This decomposition is useful in proving the following result. □

Theorem 3.2. *Let (X,τ_1,τ_2) be a bitopological space and A be a subset of X. Then*

(i) A is $(1,2)^*$-sg -closed if and only if $X_1 \cap (1,2)^*$-scl(A) $\subseteq A$.
(ii) $(1,2)^*$-pcl(A) $\subseteq X_1 \cup A$.

Proof. (i) Let $A \subseteq X$ be $(1,2)^*$-sg-closed and let $x \in X_1 \cap (1,2)^*$-scl(A). Now $\tau_{1,2}$-int($\tau_{1,2}$-cl$\{x\}$) $= \phi$ implies $\{x\}$ is $(1,2)^*$-semi-closed. If x is not in A then $A \subseteq \{x\}^c$, a $(1,2)^*$-semi-open set. Since A is $(1,2)^*$-sg-closed, $(1,2)^*$-scl(A) $\subseteq \{x\}^c$ which implies $x \notin (1,2)^*$-scl(A), a contradiction. Hence $x \in A$ and $X_1 \cap (1,2)^*$-scl(A) $\subseteq A$. Conversely let $X_1 \cap (1,2)^*$-scl(A) $\subseteq A$. Let U be any $(1,2)^*$-semi-open set containing A. It is enough if we prove that $X_2 \cap (1,2)^*$-scl(A) $\subseteq U$. Let $x \in X_2 \cap (1,2)^*$-scl(A). $\{x\}$ is $(1,2)^*$-preopen implies $\{x\} \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl$\{x\}$) $= G$. Suppose x is not in U. Then x is in U^c. Therefore $G = \tau_{1,2}$-int($\tau_{1,2}$-cl$\{x\}$) $\subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl(U^c) $\subseteq U^c$ since U^c is $(1,2)^*$-semi-closed. Then $G \cap U = \phi$ which implies $G \cap A = \phi$. This is a contradiction since, $x \in G$, a $(1,2)^*$-semi-open set and $x \in (1,2)^*$-scl(A) imply $G \cap A \neq \phi$.
(ii): Let $x \in (1,2)^*$-pcl(A). Suppose $x \notin X_1$. Then $\{x\}$ is $(1,2)^*$-preopen and thus $\{x\} \cap A \neq \phi$. This implies $x \in A$. □

Every $(1,2)^*$-sg-closed subset of a bitopological space (X,τ_1,τ_2) is $(1,2)^*$-gs-closed. The converse is not true in general.

Definition 3.2. *A bitopological space (X,τ_1,τ_2) is said to be a $(1,2)^*$-T_{gs}-space if every $(1,2)^*$-gs-closed subset of X is $(1,2)^*$-sg-closed.*

The following result characterizes the class of $(1,2)^*$-T_{gs}-spaces.

Theorem 3.3. *The following are equivalent for a bitopological space (X,τ_1,τ_2).*

(i) (X,τ_1,τ_2) is a $(1,2)^*$-T_{gs}-space.
(ii) Every singleton $\{x\}$ of X is either $\tau_{1,2}$-closed or $(1,2)^*$-preopen.

Proof. (i) ⇒ (ii) Let $x \in X_1$ and suppose that $\{x\}$ is not $\tau_{1,2}$-closed. Then $X - \{x\}$ is $(1,2)^*$-gs-closed, $(1,2)^*$-dense and $(1,2)^*$-semi-open. Since (X,τ_1,τ_2) is a $(1,2)^*$-T_{gs}-space, $X - \{x\}$ is $(1,2)^*$-sg-closed. Hence $X - \{x\} \supseteq X_1 \cap (1,2)^*$-scl($X - \{x\}$) $= X_1 \cap X = X_1$ or $x \notin X_1$, a contradiction.
(ii) ⇒ (i): Let A be $(1,2)^*$-gs-closed and let $x \in X_1 \cap (1,2)^*$-scl(A). Then $\{x\}$ is

$\tau_{1,2}$-closed. If $x \notin A$, then $A \subseteq X - \{x\}$, a $\tau_{1,2}$-open set. Since A is $(1,2)^*$-gs-closed, $(1,2)^*$-scl$(A) \subseteq X - \{x\}$, a contradiction. □

Now let us define $(1,2)^*$-extremally disconnected bitopological spaces.

Definition 3.3. *A bitopological space (X, τ_1, τ_2) is said to be $(1,2)^*$-extremally disconnected if the $\tau_{1,2}$-closure of every $\tau_{1,2}$-open subset of X is $\tau_{1,2}$-open.*

Remark 3.1. *If a bitopological space (X, τ_1, τ_2) is $(1,2)^*$-extremally disconnected, then $(1,2)^*$-$\alpha O(X, \tau_1, \tau_2) = (1,2)^*$-$SO(X, \tau_1, \tau_2)$ since $\tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A))) $= \tau_{1,2}$-cl($\tau_{1,2}$-int(A)).*

Theorem 3.4. *If $(1,2)^*$-$SO(X, \tau_1, \tau_2)$ of a bitopological space (X, τ_1, τ_2) forms a topology then (X, τ_1, τ_2) is $(1,2)^*$-extremally disconnected.*

Proof. Suppose (X, τ_1, τ_2) is not $(1,2)^*$-extremally disconnected, then there exists a $\tau_{1,2}$-open set A such that $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)) $\neq \tau_{1,2}$-cl(A). Let $x \in \tau_{1,2}$-cl$(A) - \tau_{1,2}$-int($\tau_{1,2}$-cl(A)). Let $B = \{x\} \cup \tau_{1,2}$-int($\tau_{1,2}$-cl$(A)$) and $C = (\tau_{1,2}$-int($\tau_{1,2}$-cl$(A)))^c = \tau_{1,2}$-cl($\tau_{1,2}$-int(A^c)). Now $\tau_{1,2}$-cl($\tau_{1,2}$-int(B)) $\supseteq \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(A))) $= \tau_{1,2}$-cl$(A) \supseteq \{x\}$. Also $\tau_{1,2}$-cl($\tau_{1,2}$-int(C)) $= \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A^c)))) $= \tau_{1,2}$-cl($\tau_{1,2}$-int(A^c)) $= C \supseteq \{x\}$. Thus B and C are $(1,2)^*$-semi-open sets, but $B \cap C = \{x\}$ is not $(1,2)^*$-semi-open. □

The converse of Theorem 3.4 need not be true as we see in the following example.

Example 3.1. *Let $X = \{a, b, c\}$; $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}; \tau_2 = \{\phi, \{b, c\}, X\}$;*
$\tau_{1,2}$-open sets $= \{\phi, \{a\}, \{a, b\}, \{b, c\}, X\} = (1,2)^*$-$SO(X, \tau_1, \tau_2)$;
(X, τ_1, τ_2) is $(1,2)^*$-extremally disconnected but $(1,2)^*$-$SO(X, \tau_1, \tau_2)$ does not form a topology.

Theorem 3.5. *If the intersection of any two $(1,2)^*$-sg-open sets of a bitopological space (X, τ_1, τ_2) is $(1,2)^*$-sg-open then (X, τ_1, τ_2) is $(1,2)^*$-extremally disconnected.*

Proof. Suppose (X, τ_1, τ_2) is not $(1,2)^*$-extremally disconnected. Then there is a $\tau_{1,2}$-open set A such that $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)) $\neq \tau_{1,2}$-cl(A). Let $x \in \tau_{1,2}$-cl$(A) - \tau_{1,2}$-int($\tau_{1,2}$-cl(A)). If $B = A \cup \{x\}$ and $C = (X - \tau_{1,2}$-cl$(A)) \cup \{x\}$, then B and C are $(1,2)^*$-semi-open and hence $(1,2)^*$-sg-open. By assumption $B \cap C = \{x\}$ is $(1,2)^*$-sg-open. Then $D = X - \{x\}$ is $(1,2)^*$-sg-closed. Now $\tau_{1,2}$-int($\tau_{1,2}$-cl$(\{x\})$) $\subseteq \tau_{1,2}$-cl(A). Also $x \in (\tau_{1,2}$-int($\tau_{1,2}$-cl$(A)))^c = \tau_{1,2}$-cl($\tau_{1,2}$-int(A^c)) $\subseteq A^c$ implies $\tau_{1,2}$-int($\tau_{1,2}$-cl$(\{x\})$) $\subseteq \tau_{1,2}$-int$(A^c) = (\tau_{1,2}$-cl$(A))^c$. Hence $\tau_{1,2}$-int($\tau_{1,2}$-cl$(\{x\})$) $= \phi$ and $x \in X_1$. Also $\tau_{1,2}$-int($\tau_{1,2}$-cl$((D)$) $= \tau_{1,2}$-int($\tau_{1,2}$-cl$(\{x\}^c)$) $= (\tau_{1,2}$-cl($\tau_{1,2}$-int$(\{x\})))^c = (\phi)^c = X$. Therefore $(1,2)^*$-scl$(D) = D \cup \tau_{1,2}$-int($\tau_{1,2}$-cl(D)) $= X$. Since D is $(1,2)^*$-sg-closed, $X_1 \cap (1,2)^*$-scl$(D) = X_1 \subseteq D = X - \{x\}$, a contradiction. □

Definition 3.4. *A bitopological space (X, τ_1, τ_2) is said to be a DRT - space if $int_{\tau_1} F = int_{\tau_2} F$ for every $\tau_{1,2}$-closed subset F of X.*

Remark 3.2. *If (X, τ_1, τ_2) is a DRT bitopological space, then*

(i) $(1,2)^*$-*pcl* $(A) = A \cup \tau_{1,2}$-*cl* $(\tau_{1,2}$-*int*$(A))$ *and*
$(1,2)^*$-*pint*$(A) = A \cap \tau_{1,2}$-*int* $(\tau_{1,2}$-*cl* $(A))$.
(ii) $(1,2)^*$-*spcl* $(A) = A \cup \tau_{1,2}$-*int* $(\tau_{1,2}$-*cl*$(\tau_{1,2}$-*int*$(A)))$ *and*
$(1,2)^*$-*spint* $(A) = A \cap \tau_{1,2}$-*cl*$(\tau_{1,2}$-*int*$(\tau_{1,2}$-*cl* $(A)))$.
(iii) If G *is* $\tau_{1,2}$*-open, then* $\tau_{1,2}$-*cl*$(G \cap D) \supseteq G \cap \tau_{1,2}$-*cl*$(D)$.

Theorem 3.6. *The following are equivalent in a DRT bitopological space* (X, τ_1, τ_2).
(i) (X, τ_1, τ_2) *is* $(1,2)^*$*-extremally disconnected.*
(ii) Every $(1,2)^*$*-semi-preclosed subset of* X *is* $(1,2)^*$*-preclosed.*
(iii) Every $(1,2)^*$*-sg-closed subset of* X *is* $(1,2)^*$*-preclosed.*
(iv) Every $(1,2)^*$*-semi-closed subset of* X *is* $(1,2)^*$*-preclosed.*
(v) Every $(1,2)^*$*-semi-closed subset of* X *is* $(1,2)^*$-α*-closed.*
(vi) Every $(1,2)^*$*-semi-closed subset of* X *is* $(1,2)^*$-$g\alpha$*-closed.*

Proof. (i)⇒(ii) If A is $(1,2)^*$-semi-preclosed, then $A = (1,2)^*\text{-spcl}(A) = A \cup \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A))) = A \cup \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A))$ since X is $(1,2)^*$-extremally disconnected. Hence $A = (1,2)^*\text{-pcl}(A)$ or A is $(1,2)^*$-preclosed.
(ii)⇒(iii) Let A be $(1,2)^*$-sg-closed. It is enough to prove that A is $(1,2)^*$-semi-preclosed. Let $x \in (1,2)^*\text{-spcl}(A)$. Case(i): $\{x\}$ is $(1,2)^*$-preopen. Then $\{x\}$ is $(1,2)^*$-semi-preopen and since $x \in (1,2)^*\text{-spcl}(A)$, $\{x\} \cap A \neq \phi$. Hence $x \in A$. Case(ii): $\{x\}$ is nowhere dense. Then $\{x\}$ is $(1,2)^*$-semi-closed which implies $X - \{x\}$ is $(1,2)^*$-semi-open. Assume that $x \notin A$. Then $A \subseteq X - \{x\}$ and A is $(1,2)^*$-sg-closed imply $(1,2)^*\text{-spcl}(A) \subseteq (1,2)^*\text{-scl}(A) \subseteq X - \{x\}$. Hence $x \notin (1,2)^*\text{-spcl}(A)$, a contradiction. Therefore $x \in A$. Thus in both the cases , $A = (1,2)^*\text{-spcl}(A)$ or A is $(1,2)^*$-semi-preclosed.
(iii)⇒(iv), (iv)→(v), (v)→(vi) are obvious.
(iv)⇒(i): Let A be $\tau_{1,2}$-open. Consider $B = \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)))$. B is $(1,2)^*$-regular open and therefore $(1,2)^*$-semi-closed and $(1,2)^*$-α-open. By (vi) B is $(1,2)^*$-gα-closed which implies $B = (1,2)^*\text{-}\alpha\text{cl}(B) = \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(B)))$. Hence $\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A))) = \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A))$ or $\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A)) = \tau_{1,2}\text{-cl}(A)$. Hence (X, τ_1, τ_2) is $(1,2)^*$-extremally disconnected. □

4. $(1,2)^*$-Submaximality

Now let us define $(1,2)^*$-submaximal bitopological spaces.

Definition 4.1. *A bitopological space* (X, τ_1, τ_2) *is said to be* $(1,2)^*$*-submaximal (resp.* $(1,2)^*$*-g-submaximal) if every* $(1,2)^*$*-dense subset of* X *is* $\tau_{1,2}$*-open (resp.* $(1,2)^*$*-g-open).*

Remark 4.1. *Every* $(1,2)^*$*-submaximal space is* $(1,2)^*$*-g-submaximal but not conversely.*

Example 4.1. *Let* $X = \{a, b, c\}$ *;* $\tau_1 = \{\phi, \{a\}, X\}$;
$\tau_2 = \{\phi, \{b, c\}, X\}$*;* $\tau_{1,2}$*-open sets* $= \{\phi, \{a\}, \{b, c\}, X\}$*;*
(X, τ_1, τ_2) *is* $(1,2)^*$*-g-submaximal but not* $(1,2)^*$*-submaximal.*

Definition 4.2. *A bitopological space* (X, τ_1, τ_2) *is said to be* $(1,2)^*$*-sg-submaximal (resp.* $(1,2)^*$-α*-submaximal) if every* $(1,2)^*$*-dense subset of* X *is* $(1,2)^*$*-sg-open (resp.* $(1,2)^*$-α*-open).*

5. Applications

Now let us see some applications of $(1,2)^*$-extremally disconnectedness and $(1,2)^*$-submaximality in bitopological spaces.

Theorem 5.1. *Let (X,τ_1,τ_2) be a DRT bitopological space in which every $(1,2)^*$-semi-preclosed set is $(1,2)^*$-gα-closed. Then $(X,\tau_1^\alpha,\tau_2^\alpha)$ is $(1,2)^*$-extremally disconnected and $(1,2)^*$-g-submaximal.*

Proof. If every $(1,2)^*$-semi-preclosed subset of X is $(1,2)^*$-gα-closed, then every $(1,2)^*$-semi-closed subset of X is $(1,2)^*$-gα-closed and hence by Theorem 3.6, (X,τ_1,τ_2) is $(1,2)^*$-extremally disconnected. Now $(1,2)^*$-αcl$((1,2)^*$-αint$(A))$ = $\tau_{1,2}$-cl$((1,2)^*$-αint$(A)) = \tau_{1,2}$-cl$(\tau_{1,2}$-int$((1,2)^*$-αint$(A)))$. Also $(1,2)^*$-αint$((1,2)^*$-αcl$(A)) = \tau_{1,2}$-int$((1,2)^*$-αcl$(A)) = \tau_{1,2}$-int$(\tau_{1,2}$-cl$((1,2)^*$-αcl$(A)))$. Hence $(1,2)^*$-αint$((1,2)^*$-αcl$((1,2)^*$-αint$(A))) = \tau_{1,2}$-int$((1,2)^*$-αcl$((1,2)^*$-αint$(A)))$ = $\tau_{1,2}$-int$(\tau_{1,2}$-cl$((1,2)^*$-αint$(A))) = \tau_{1,2}$-int$(\tau_{1,2}$-cl$(\tau_{1,2}$-int$((1,2)^*$-αint$(A)))) = \tau_{1,2}$-cl$(\tau_{1,2}$-int$((1,2)^*$-αint$(A))) = (1,2)^*$-αcl$((1,2)^*$-αint$A))$ for every subset A of X. Therefore $(X,\tau_1^\alpha,\tau_2^\alpha)$ is $(1,2)^*$-extremally disconnected. Let $A\subseteq X$ be a dense subset in $(X,\tau_1^\alpha,\tau_2^\alpha)$. Then $(1,2)^*$-αcl$(A)=X$. Since $\tau_{1,2}$-cl$(A\supseteq(1,2)^*$-αcl(A), $\tau_{1,2}$-cl$(A)=X$. This implies $\tau_{1,2}$-cl$(\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A)))=X$ and therefore A is $(1,2)^*$-semi-preopen or $X-A$ is $(1,2)^*$-semi-preclosed. So A^c is $(1,2)^*$-gα-closed and A is $(1,2)^*$-g-open in $(X,\tau_1^\alpha,\tau_2^\alpha)$. □

Lemma 5.1. *(X,τ_1,τ_2) and $(X,\tau_1^\alpha,\tau_2^\alpha)$ share the classes of dense subsets.*

Proof. It has been proved in Theorem 5.1 that, if A is dense in $(X,\tau_1^\alpha,\tau_2^\alpha)$ then A is dense in (X,τ_1,τ_2). Conversely let A be dense in (X,τ_1,τ_2). Then $\tau_{1,2}$-cl(A) $=X$ which implies $\tau_{1,2}$-cl$(\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A)))=X$. Therefore $(1,2)^*$-αcl(A) = $A\cup\tau_{1,2}$-cl$(\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A)))=X$ or A is dense in $(X,\tau_1^\alpha,\tau_2^\alpha)$. □

Remark 5.1. *Since (X,τ_1,τ_2) and $(X,\tau_1^\alpha,\tau_2^\alpha)$ share the classes of dense subsets (X,τ_1,τ_2) is $(1,2)^*$-α-submaximal if and only if $(X,\tau_1^\alpha,\tau_2^\alpha)$ is $(1,2)^*$-submaximal.*

Lemma 5.2. *(X,τ_1,τ_2) and $(X,\tau_1^\alpha,\tau_2^\alpha)$ share the classes of $(1,2)^*$-sg-open subsets.*

Proof. Let A be any $(1,2)^*$-sg-closed set in (X,τ_1,τ_2) and U be any $(1,2)^*$-semi-open set in $(X,\tau_1^\alpha,\tau_2^\alpha)$ containing A. Then $U\subseteq(1,2)^*$-αcl$((1,2)^*$-αint$(U))$ = $\tau_{1,2}$-cl$(\tau_{1,2}$-int$((1,2)^*$-αint$(U)))\subseteq\tau_{1,2}$-cl$(\tau_{1,2}$-int$(U))$, which implies U is $(1,2)^*$-semi-open in (X,τ_1,τ_2). Since A is $(1,2)^*$-sg-closed set in (X,τ_1,τ_2), $(1,2)^*$-scl$(A)\subseteq U$. Now $(1,2)^*$-αint$((1,2)^*$-αcl$(A))=\tau_{1,2}$-int$(\tau_{1,2}$-cl$((1,2)^*$-αcl$(A)))$ $=\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A\cup\tau_{1,2}$-cl$(\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A)))))\subseteq\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A))\subseteq(1,2)^*$-scl$(A)\subseteq U$. Hence A is $(1,2)^*$-sg-closed in $(X,\tau_1^\alpha,\tau_2^\alpha)$. Conversely let A be $(1,2)^*$-sg-closed in $(X,\tau_1^\alpha,\tau_2^\alpha)$ and U be any $(1,2)^*$-semi-open set in (X,τ_1,τ_2) containing A. Then $(1,2)^*$-αcl$((1,2)^*$-αint$(U))=\tau_{1,2}$-cl$(\tau_{1,2}$-int$((1,2)^*$-αint$(U)))$ = $\tau_{1,2}$-cl$(\tau_{1,2}$-int$(U\cap\tau_{1,2}$-int$(\tau_{1,2}$-cl$(\tau_{1,2}$-int$(U)))))\supseteq\tau_{1,2}$-cl$(\tau_{1,2}$-int$(U))\supseteq U$. Hence U is $(1,2)^*$-semi-open set in $(X,\tau_1^\alpha,\tau_2^\alpha)$. Since A is $(1,2)^*$-sg-closed in $(X,\tau_1^\alpha,\tau_2^\alpha)$, $(1,2)^*$-scl(A) in $(X,\tau_1^\alpha,\tau_2^\alpha)$ is contained in U. That is $A\cup(1,2)^*$-αint$((1,2)^*$-αcl$(A))\subseteq U$. Therefore $\tau_{1,2}$-int$(\tau_{1,2}$-cl$(A))\subseteq\tau_{1,2}$-int$(\tau_{1,2}$-cl$((1,2)^*$-αcl$(A))$ =

$(1,2)^*$-αint($(1,2)^*$-αcl(A)) $\subseteq U$. This implies $(1,2)^*$-scl$(A) = A \cup \tau_{1,2}$-int($\tau_{1,2}$-cl(A)) $\subseteq U$. Hence A is $(1,2)^*$-sg-closed in (X, τ_1, τ_2). □

Theorem 5.2. *A bitopological space (X, τ_1, τ_2) is $(1,2)^*$-sg-submaximal if and only if the bitopological space $(X, \tau_1^\alpha, \tau_2^\alpha)$ is $(1,2)^*$-sg-submaximal.*

Proof follows from Lemmas 5.1 and 5.2.

Now let us see some properties of the $(1,2)^*$-preopen sets in a $(1,2)^*$-submaximal space. We first prove some simple characterisation of $(1,2)^*$-preopen sets in a DRT bitopological space.

Theorem 5.3. *For any subset S of a DRT bitopological space (X, τ_1, τ_2) the following are equivalent*

(i) $S \in (1,2)^*$-$PO(X)$.
(ii) There is a $(1,2)^*$-regular open set $G \subseteq X$ such that $S \subseteq G$ and $\tau_{1,2}$-cl$(S) = \tau_{1,2}$-cl(G).
(iii) S is the intersection of a $(1,2)^*$-regular open set and a $(1,2)^*$-dense set.
(iv) S is the intersection of a $\tau_{1,2}$- open set and a $(1,2)^*$-dense set.

Proof. (i) ⇒(ii) Let $S \in (1,2)^*$-PO(X) and $G = \tau_{1,2}$-int($\tau_{1,2}$-cl(S)). Then G is $(1,2)^*$-regular open with $S \subseteq G$. Now $\tau_{1,2}$-cl$(G) = \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(S))) $\subseteq \tau_{1,2}$-cl$(S) \subseteq \tau_{1,2}$-cl(G). Hence $\tau_{1,2}$-cl$(G) = \tau_{1,2}$-cl(S).
(ii) ⇒(iii) Let $D = S \cup (X - G)$. Then $\tau_{1,2}$-cl$(D) \supseteq \tau_{1,2}$-cl$(S) \cup \tau_{1,2}$-cl$(G^c) \supseteq \tau_{1,2}$-cl$(S) \cup G^c = \tau_{1,2}$-cl$(G) \cup G^c \supseteq X$. Therefore $\tau_{1,2}$-cl$(D) = X$ or D is $(1,2)^*$-dense in X. Also $G \cap D = S$.
(iii) ⇒(iv): It follows since every $(1,2)^*$-regular open set is $\tau_{1,2}$-open. (iv) ⇒(i): Suppose $S = G \cap D$ with G, $\tau_{1,2}$-open and D, $(1,2)^*$-dense. Then $\tau_{1,2}$-cl$(S) = \tau_{1,2}$-cl$(G \cap D) \supseteq G \cap \tau_{1,2}$-cl$(D) \supseteq G \cap X = G$. Hence $\tau_{1,2}$-cl$(S) \supseteq \tau_{1,2}$-cl$(G) \supseteq \tau_{1,2}$-cl(S) or $\tau_{1,2}$-cl$(S) = \tau_{1,2}$-cl(G). Hence $S \subseteq G \subseteq \tau_{1,2}$-cl$(G) \subseteq \tau_{1,2}$-cl$(S)$ which imply $S \subseteq G = \tau_{1,2}$-int$(G) \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl$(S)$). Hence S is $(1,2)^*$-preopen. □

Theorem 5.4. *Let (X, τ_1, τ_2) be a bitopological space. Then*

(i) $(1,2)^*$-$SO(X, \tau_1, \tau_2) = (1,2)^*$-$SO(X, \tau_1^\alpha, \tau_2^\alpha)$.
(ii) $(1,2)^*$-$PO(X, \tau_1, \tau_2) = (1,2)^*$-$PO(X, \tau_1^\alpha, \tau_2^\alpha)$.
(iii) $(1,2)^*$-$\alpha O(X, \tau_1, \tau_2) = (1,2)^*$-$\alpha O(X, \tau_1^\alpha, \tau_2^\alpha)$.

Proof. (i) is proved in Lemma 5.2.
(ii) Let $A \in (1,2)^*$-PO(X, τ_1, τ_2). Now $(1,2)^*$-αint($(1,2)^*$-αcl(A)) $= \tau_{1,2}$-int($\tau_{1,2}$-cl($(1,2)^*$-αcl(A))) $\supseteq \tau_{1,2}$-int($\tau_{1,2}$-cl(A)) $\supseteq A$. Therefore $A \in (1,2)^*$-PO$(X, \tau_1^\alpha, \tau_2^\alpha)$. Let $A \in (1,2)^*$-PO$(X, \tau_1^\alpha, \tau_2^\alpha)$. Then $A \subseteq (1,2)^*$-αint($(1,2)^*$-αcl(A)) $\subseteq (1,2)^*$-αint($\tau_{1,2}$-cl(A)) $= \tau_{1,2}$-cl$(A) \cap \tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(A)))) $= \tau_{1,2}$-cl$(A) \cap \tau_{1,2}$-int($\tau_{1,2}$-cl(A)) $= \tau_{1,2}$-int($\tau_{1,2}$-cl(A)). Therefore $A \in (1,2)^*$-PO(X, τ_1, τ_2). Hence $(1,2)^*$-PO$(X, \tau_1, \tau_2) = (1,2)^*$-PO$(X, \tau_1^\alpha, \tau_2^\alpha)$.
(iii) $(1,2)^*$-$\alpha O(X, \tau_1, \tau_2) = (1,2)^*$-SO$(X, \tau_1, \tau_2) \cap (1,2)^*$-PO$(X, \tau_1, \tau_2) = (1,2)^*$-SO$(X, \tau_1^\alpha, \tau_2^\alpha) \cap (1,2)^*$-PO$(X, \tau_1^\alpha, \tau_2^\alpha) = (1,2)^*$-$\alpha O(X, \tau_1^\alpha, \tau_2^\alpha)$. □

Theorem 5.5. *Let (X, τ_1, τ_2) be a bitopological space, $S \in (1,2)^*$-$PO(X, \tau_1, \tau_2)$ and $x \in \tau_{1,2}$-cl$(S) - \tau_{1,2}$-int($\tau_{1,2}$-cl(S)). Then $S \cup \{x\} \notin (1,2)^*$-$PO(X, \tau_1, \tau_2)$. In* *particular if G is $(1,2)^*$-regular open in (X,τ_1,τ_2) and $x \in \tau_{1,2}$-cl(G) $- G$, then $G \cup \{x\} \notin (1,2)^*$-PO$(X,\tau_1,\tau_2)$.*

Proof. Since $x \in \tau_{1,2}$-cl(S), $\tau_{1,2}$-cl$(S) \supseteq \tau_{1,2}$-cl$(S \cup \{x\}) \supseteq \tau_{1,2}$-cl$(S)$. Therefore $\tau_{1,2}$-cl$(S) = \tau_{1,2}$-cl$(S \cup \{x\})$ which implies $\tau_{1,2}$-int$(\tau_{1,2}$-cl$(S \cup \{x\})) = \tau_{1,2}$-int$(\tau_{1,2}$-cl$(S))$. Since $x \notin \tau_{1,2}$-cl(S), $S \cup \{x\}) \nsubseteq \tau_{1,2}$-int$(\tau_{1,2}$-cl$(S \cup \{x\}))$ and $S \cup \{x\} \notin (1,2)^*$-PO$(X,\tau_1,\tau_2)$. □

Theorem 5.6. *If (X,τ_1,τ_2) is a $(1,2)^*$-submaximal DRT bitopological space, then any $(1,2)^*$-preopen set is the intersection of two $\tau_{1,2}$-open sets.*

Proof. By Theorem 5.3, if A is $(1,2)^*$-preopen then $A = G \cap D$ where G is $\tau_{1,2}$-open and D is $(1,2)^*$-dense in (X,τ_1,τ_2). Since (X,τ_1,τ_2) is $(1,2)^*$-submaximal, D is $\tau_{1,2}$-open. Hence A is the intersection of two $\tau_{1,2}$-open sets. □

Theorem 5.7. *If in a bitopological space (X,τ_1,τ_2), $(1,2)^*$-PO$(X,\tau_1,\tau_2) = \tau_{1,2}$-open sets, then (X,τ_1,τ_2) is $(1,2)^*$-submaximal.*

Proof. Let $(1,2)^*$-PO$(X,\tau_1,\tau_2) = \tau_{1,2}$-open sets. Let D be $(1,2)^*$-dense in (X,τ_1,τ_2). Then $\tau_{1,2}$-cl$(D) = X$. This implies $\tau_{1,2}$-int$(\tau_{1,2}$-cl$(D)) = X \supseteq D$. Therefore D is $(1,2)^*$-preopen which implies D is $\tau_{1,2}$-open. Hence (X,τ_1,τ_2) is $(1,2)^*$-submaximal.
Converse of Theorem 5.7 is not true in general. □

Example 5.1. *Let $X = \{a,b,c,d\}$; $\tau_1 = \{\phi,\{d\},\{a,b,d\},X\}$*
$\tau_2 = \{\phi,\{c\},\{c,d\},\{b,c,d\},\{a,c,d\},X\}$;
$\tau_{1,2}$-open sets $= \{\phi,\{d\},\{a,b,d\},\{c\},\{c,d\},\{b,c,d\},\{a,c,d\},X\}$.
$\tau_{1,2}$-closed sets $= \{\phi,\{a,b,c\},\{c\},\{a,b,d\},\{a,b\},\{a\},\{b\},X\}$.
(X,τ_1,τ_2) is $(1,2)^*$-submaximal but $(1,2)^*$-PO$(X,\tau_1,\tau_2) \neq \tau_{1,2}$-open sets since $\{b,d\}$ is $(1,2)^*$-preopen but not $\tau_{1,2}$-open.

References

[1] Andrijevic.D, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.
[2] Bhattacharya,P. and Lahiri,B.K, Semi-generalised closed sets in topology, Indian J. Math., 29(1987),375-382
[3] Jiling Cao, Maximillion Ganster and Ivan Reily, On sg-closed sets and gα closed sets, Mem. Fac. Kochi. Univ. (Math.), (20)(1999), 1-5.
[4] Levine,N, Semi-open sets and semi-continuity in topological spaces, Amer.Math. Monthly, 70 (1963) , 36-41.
[5] Levine,N, Generalised closed sets in topology, Rend. circ. Mat. Palermo, 19(1970), 89-96.
[6] Mashhour,A.S., Abd El-Monsef,M.E. and El Deeb,S, NOn pre continuous and weak pre continuous mappings,Proc.Math.Phys. Soc. Egypt.,53(1982),47-53.
[7] Njastad,O, On some classes of nearly open sets, Pacific J.Math.15(1965),961-970
[8] Ravi,O. and Lellis Thivagar,M, On Stronger forms of $(1,2)^*$-quotient mappings in bitopological spaces,Internat. J. Math. Game Theory and Algebra,14(2004),no.6,481-492

[1]School of Mathematics, Madurai Kamaraj University, Madurai,Tamilnadu, India
E-mail address: mlthivagar@yahoo.co.in

[2]Department of Mathematics, Lady Doak College, Madurai-625002,
Tamil Nadu, India
E-mail address: nirmala_mariappan@yahoo.com