Математички Билтен
Книга 9-10 (XXXV-XXXVI)
1985-1986 (49-55)
Скопје, Југославија

TWO THEOREMS FROM DIFFERENTIAL CALCULUS*)

Jovan V. Malešević

The paper presents two theorems from differential calculus, in some sense the analogons of Cauchy's theorem, as the generalizations of the corresponding theorems from paper [1].

Theorem 1. Let the function $f(x)$ and $\phi(x)$ be continuous on the segment $[a,b]$, have the corresponding unilateral derivatives at the points a and b, and let be

$$\left[f'(a)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(a)\right]\left[f'(b)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(b)\right]>0 \quad (\phi(b)\neq\phi(a)) \tag{1}$$

Then the equation

$$G_r(x)\equiv f(x)-f(r)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}[\phi(x)-\phi(r)]=0 \tag{2}$$

$r=a$ or $r=b$, has at least one zero in the interval (a,b) at which function $G_r(x)$ alters the sign.

Proof. From condition (1) either

$$f'(a)>\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(a)\ \wedge\ f'(b)>\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(b), \tag{3}$$

or

$$f'(a)<\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(a)\ \wedge\ f'(b)<\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(b) \tag{4}$$

follows.

In case (3), writting

$$\Phi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}[\phi(x)-\phi(a)] \quad \left(\equiv f(x)-f(b)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}[\phi(x)-\phi(b)]\right) \tag{5}$$

we get

$$\Phi'_+(a)=f'(a)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(a)\ \wedge \quad \Phi'_-(b)=f'(b)-\frac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(b)>0, \tag{6}$$

---

*) Presented on January 22, 1982 at the Institute of Mathematics in Belgrade, Yugoslavia.

and so there exist numbers η_1 and η_2 such that

$$a < \eta_1 < \eta_2 < b \wedge \Phi(\eta_1) > 0, \ \Phi(\eta_2) < 0.$$

Because of continuity of the function $\Phi(x)$ on the segment $[a,b]$, there exists $\eta \in (\eta_1, \eta_2) \subset (a,b)$ such that $\Phi(\eta)=0$, i.e.

$$G_r(\eta) \equiv f(\eta)-f(r) - \frac{f(b)-f(a)}{\phi(b)-\phi(a)}[\phi(\eta)-\phi(r)] = 0 \quad (r=a \text{ or } r=b).$$

In a similar way, the same conclusion is obtained in case (4). Thus the theorem is entirely proved.

For $\phi(x) = x$ from the above theorem, theorem 1 from paper [1] follows.

For

$$\phi(r) = f(r) \quad (r=a \text{ and } r=b), \tag{7}$$

from the above theorem it follows that, under the given condition, the graphs of the functions $y=f(x)$ and $y=\phi(x)$ intersect above the interval (a,b) at least at one point $C[\eta, f(\eta)=\phi(\eta)]$, $\eta \in (a,b)$. Then condition (1) reduces to the form

$$[f'(a) - \phi'(a)][f'(b) - \phi'(b)] > 0^{*)}. \tag{1'}$$

<u>Definition</u>. Let (f,ϕ) be a pair of continuous functions on $[a,b]$ having the unilateral derivatives at the points a and b and the quotiens $\frac{f'(a)}{\phi'(a)}$ and $\frac{f'(b)}{\phi'(b)}$ have the meaning

$$\lim_{x \to a_+} \frac{f(x)-f(a)}{\phi(x)-\phi(a)}, \quad \lim_{x \to b_-} \frac{f(x)-f(b)}{\phi(x)-\phi(b)} \tag{8}$$

respectively. If

$$\begin{matrix}\min. \\ \text{or} \\ \max.\end{matrix}\left\{\frac{f'(a)}{\phi'(a)}, \frac{f'(b)}{\phi'(b)}, \frac{f(b)-f(a)}{\phi(b)-\phi(a)}\right. = \frac{f(b)-f(a)}{\phi(b)-\phi(a)} \ (\phi(b) \neq \phi(a)), \tag{9}$$

then we say the pair (f,ϕ) is <u>regular</u>. If, in addition, both $\frac{f'(a)}{\phi'(a)}$, $\frac{f'(b)}{\phi'(b)}$ are different from $\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$, we say the pair (f,ϕ) is <u>strictly regular</u>.

Note that if

$$0 < \phi'(a) \cdot \phi'(b) < +\infty, \tag{10}$$

then Theorem 1 is valid for a strictly regular pair (f,ϕ). This

---

*) In case of the equation $f(x)-f(r)-[\phi(x)-\phi(r)]=0$, under $f(b)-f(a)=\phi(b)-\phi(a) \neq 0$, the above condition of existence of a zero in the interval (a,b) is also sufficient.

implies that the condition 3″) in the Theorem 4 from [2], under $g'(a)g'(b) > 0$, has the meaning that the set Ω is nonempty.

Corollary 1. If the functions $f(x)$ and $\phi(x)$ on the segment $[a,b]$ fulfill the conditions of theorem 1, have the derivatives in the interval (a,b) and

$$\phi'(x) \neq 0, \quad \forall x \in (a,b)^{*)}; \tag{11}$$

then there are at least two numbers $\xi_1, \xi_2 \in (a,b)$ such that

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(\xi_i)}{\phi'(\xi_i)}, \quad i=1,2 \tag{12}$$

– specification of Cauchy's theorem for the given hypotheses.

Indeed, by Theorem 1, the equation

$$\Phi(x) \equiv [\phi(b)-\phi(a)][f(x)-f(a)]-[f(b)-f(a)][\phi(x)-\phi(a)]=0 \tag{13}$$

has at least one zero $\eta \in (a,b)$ and from

$$\Phi(a) = \Phi(b) = \Phi(\eta) = 0,$$

the statement follows.

Let us remark that the statement of corollary also holds for the pairs of strictly regular functions (f,ϕ) which have the derivatives in the interval (a,b), and for which conditions (10) and (11) hold.

Note that for $\phi(x)=x$ corollary 1 contains corollary 1 from paper [1].

Corollary 2. If the functions $f(x)$ and $\phi(x)$ on the segment $[a,b]$ fulfill the conditions of Theorem 1, have the derivatives in the interval (a,b) and

$$\phi'(x) \neq 0, \quad \forall x \in (a,b)^{**)}; \tag{14}$$

then there exist at least two numbers $\xi_1, \xi_2 \in (a,b)$ such that

$$\frac{f'(\xi_1)}{\phi'(\xi_1)} = \frac{f(\xi_1)-f(a)}{\phi(\xi_1)-\phi(a)} \wedge \frac{f'(\xi_2)}{\phi'(\xi_2)} = \frac{f(\xi_2)-f(b)}{\phi(\xi_2)-\phi(b)}, \tag{15}$$

where $\xi_1 \in (\eta_1, b)$, $\xi_2 \in (a, \eta_2)$, η_2 – being a zero of the equation

---

*) The condition $\phi(b) \neq \phi(a)$ now is the result of condition (11).

**) It follows: $\phi(b) \neq \phi(a) \wedge \phi(x) \neq \phi(r)$, $\forall x \in (a,b)$, $(r=a$ and $r=b)$.

$G_a(x) = 0$ on the interval (a,b), η_2 - a zero of the equation $G_b(x) = 0$ on the same interval. The functions $G_a(x)$ and $G_b(x)$ are given by

$$G_a(x)=\begin{cases} \dfrac{f(x)-f(a)}{\phi(x)-\phi(a)} - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}, & x\in(a,b], \\ \dfrac{f'(a)}{\phi'(a)} - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}, & x=a; \end{cases} \tag{16}$$

$$G_b(x)=\begin{cases} \dfrac{f(x)-f(b)}{\phi(x)-\phi(b)} - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}, & x\in[a,b), \\ \dfrac{f'(b)}{\phi'(b)} - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}, & x=b. \end{cases} \tag{17}$$

Indeed, for $x\in(a,b]$, by Theorem 1 and (16)

$$G_a(\eta_1) = G_a(b) = 0, \ \eta_1\in(a,b)$$

follows; for $x\in[a,b)$, by Theorem 1 and (17)

$$G_b(\eta_2) = G_b(a) = 0, \ \eta_2\in(a,b)$$

follows. Now applying the Rolle's theorem we obtain

$$G_a'(\xi_1) = 0, \ \xi_1\in(\eta_1,b) \quad G_b'(\xi_2) = 0, \ \xi_2\in(a,\eta_2),$$

respectively, and hence the corresponding conclusions in (15) follow.

Let us remark that the statement of corollary 2 also holds for the pairs of strictly regular functions (f,ϕ) which have the derivatives in the interval (a,b) and for which conditions (10) and (14) hold.

For $\phi(x)=x$ corollary 2 contains corollary 4 from paper [1].

<u>Theorem 2</u>. Let the functions $f(x)$ and $\phi(x)$ be continuons on the segment $[a,b]$, have the corresponding unilateral derivatives at the points a and b and quotients $\frac{f'(a)}{\phi'(a)}, \frac{f'(b)}{\phi'(b)}$ are defined*) and at least one of the relations

$$\frac{f'(r)}{\phi'(r)} \neq \frac{f(b)-f(a)}{\phi(b)-\phi(a)}, \quad r=a \text{ or } r=b, \tag{18}$$

holds where it is supposed that

$$\phi(b) \neq \phi(a) \wedge \phi(x) \neq \phi(r), \ \forall x\in(a,b) \tag{19}$$

($r=a$ or $r=b$). If the value c is between

---

*) The relations (8).

$$\frac{f'(r)}{\phi'(r)} \text{ and } \frac{f(b)-f(a)}{\phi(b)-\phi(a)} \tag{20}$$

(r=a or r=b), then corresponding equation

$$F_r(x) \equiv \frac{f(x)-f(r)}{\phi(x)-\phi(r)} - c = 0, \quad r=a \text{ or } r=b, \tag{21}$$

has at least one zero $\eta \in (a,b)$ in which the function $F_r(x)$ alters the sign.

Proof. Let, for example,

$$\frac{f'(a)}{\phi'(a)} > c > \frac{f(b)-f(a)}{\phi(b)-\phi(a)}. \tag{22}$$

Then for function

$$F_a(x) = \begin{cases} \dfrac{f(x)-f(a)}{\phi(x)-\phi(a)} - c, & a < x \le b, \\ \dfrac{f'(a)}{\phi'(a)} - c, & x = a; \end{cases} \tag{23}$$

holds

$$F_a(a+0) = \frac{f'(a)}{\phi'(a)} - c > 0.$$

Therefore, there exists a number $\alpha \in (a,b)$ such that $F_a(\alpha) > 0$. Since

$$F_a(b) = \frac{f(b)-f(a)}{\phi(b)-\phi(a)} - c < 0,$$

and the function $F_a(x)$ is continuous on the segment $[\alpha,b]$, there exists a number $\eta \in (\alpha,b) \subset (a,b)$ such that $F_a(\eta) = 0$, that is

$$\frac{f(\eta)-f(a)}{\phi(\eta)-\phi(a)} - c = 0.$$

In the other cases, one proceeds similarly.

For $\phi(x) = x$ the above theorem contains Theorem 2 from paper [1].

Corollary 3. Let the functions $f(x)$ and $\phi(x)$ be continuous on the segment $[a,b]$, have the corresponding unilateral derivatives at the points a and b, and derivatives in the interval (a,b), and

$$\phi'(x) \neq 0, \quad \forall x \in (a,b). \tag{24}$$

If

$$\frac{f'(a)}{\phi'(a)} \neq \frac{f'(b)}{\phi'(b)} \; ^{*)} \tag{25}$$

and the value c is between $\frac{f'(a)}{\phi'(a)}$ and $\frac{f'(b)}{\phi'(b)}$, then there exists a number $\xi \in (a,b)$ such that

---

*) This supposes the existence of the quotients $\frac{f'(a)}{\phi'(a)}$ and $\frac{f'(b)}{\phi'(b)}$.

$$\frac{f'(\bar{\xi})}{\phi'(\bar{\xi})} = c. \tag{26}$$

Proof. Since $\frac{f'(a)}{\phi'(a)} \neq \frac{f'(b)}{\phi'(b)}$ and the number c is between $\frac{f'(a)}{\phi'(a)}$ and $\frac{f'(b)}{\phi'(b)}$, we have

$$\frac{f'(r)}{\phi'(r)} = \frac{f(b)-f(a)}{\phi(b)-\phi(a)} \qquad (\phi(b)\neq\phi(a)), \tag{27}$$

at least for one r, $r=a$ or $r=b$. For such r, the numer c is between $\frac{f'(r)}{\phi'(r)}$ and $\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$ or $c=\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$.

In the former case, by Theorem 2, there is $\eta_r\in(a,b)$ such that η_r is a zero of equation

$$\Phi_r(x) \equiv f(x)-f(r) - c[\phi(x)-\phi(r)] = 0. \tag{28}$$

Now from

$$\Phi_r(r) = 0 \wedge \Phi_r(\eta_r) = 0$$

it follows that between r and η_r there exists such $\bar{\xi}_r$ that $\Phi'(\bar{\xi}_r) = 0$, i.e. $\frac{f'(\bar{\xi}_r)}{\phi'(\bar{\xi}_r)} = c$ *).

When c equals to $\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$, the existence of the point follows by the Cauchy's theorem.

For $\phi(x) = x$ the above corollary contains the well-known theorem of Darboux from the classic analysis [3].

Under assumption that relation (27) holds at least for one r, $r=a$ or $r=b$, corollary 3 gives, in some sense, an extension of the Cauchy's formula from the value $c=\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$ to the value c between $\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$ and $\frac{f'(r)}{\phi'(r)}$, $r=a$ or $r=b$ **).

If

$$\frac{f'(r)}{\phi'(r)} = \frac{f(b)-f(a)}{\phi(b)-\phi(a)}, \quad r=a \text{ and } r=b, \tag{29}$$

the above extension does not exist and only the Cauchy's formula remains.

Specially, for a strictly regular pair, we have

$$\frac{f'(r)}{\phi'(r)} \neq \frac{f(b)-f(a)}{\phi(b)-\phi(a)}, \quad r=a \text{ and } r=b, \tag{30}$$

and then the mentioned extension holds for all values c between

---

*) In this case a more precise result than usually stated is obtained: $\bar{\xi}_r$ is between r and η_r, $\eta_r\in(a,b)$.

**) For $\phi(x)=x$ the Darboux theorem can be considered as an extension of the Lagrange's formula.

$$\min_{x=a,b} \frac{f''(x)}{\phi''(x)} \quad \text{and} \quad \frac{f(b)-f(a)}{\phi(b)-\phi(a)}$$

and for all values c between

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} \text{ and } \max_{x=a,b} \frac{f''(x)}{\phi''(x)}$$

respectively.

REFERENCES

[1] J.V. Malešević: On a theorem of G. Darboux and a theorem of D. Traian (Serbo-Croatian), Mat. Vesnik 4 (17) (32), (1980), N° 3

[2] J.V. Malešević: A certain generalization of Taylor's formula (Serbo-Croatian), Mat. Vesnik 12 (27) (1975), N° 4

[3] М.К. Гребенча и С.И. Новоселов: Курс математического анализа, часть I, Москва, 1960, 307–308

DVA STAVA IZ DIFERENCIJALNOG RAČUNA

Jovan V. Malešević

Rezime

U radu su, kao novi rezultati dati stavovi 1 i 2, analogni teoremi Cauchy-a iz matematičke analize, i posledice stavova 1 i 2. Stavovi 1 i 2, sa posledicama, uopštavaju stavove 1 i 2 i neke njihove posledice iz rada [1].