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ON THE UNIFORM APPROXIMATION OF AN INNER FUNCTION

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Abstract

The inner function $\exp\left(-\frac{1+z}{1-z}\right)$ we approximate uniformly in the unit disk by interpolating Blaschke products.

Let D be the unit disk in the plane. If f is a bounded holomophic function in the unit disk then

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

exist almost everywhere on ∂D and is called radial boundury function of f. We call the holomorphic function $I(z)$, $z \in D$, an inner function if its radial boundury function $I^*(e^{i\theta})$ satisfy $|I^*(e^{i\theta}| = 1$, e.e. on ∂D.

A Blaschke product in the unit disk is the holomorphic function

$$b(z) = z^n \prod_{k=1}^{\infty} \frac{\lambda_k - z}{1 - \overline{\lambda}_k z} \mu_k, \quad \lambda_k \neq 0, \quad k = 1, 2, \ldots, \quad \sum_{k=1}^{\infty} (1 - |\lambda_k|) < \infty,$$

where μ_k is a convergence factor, for example $\mu_k = \frac{|\lambda_k|}{\lambda_k}$, $k = 1, 2, \ldots$. The numbes λ_k, $k = 1, 2, \ldots$ are zeros of the Blaschke product above. A finite Blaschke product is a product with finite many zeros. Every Blaschke product is an inner function in the unit disk.

A sequence $(\lambda_k)_{k=1}^{\infty} \subset D$ is called a H^∞ interpolating sequence if for every bounded sequence $(c_k)_{k=1}^{\infty}$ there exists bounded holomorphic function f such that $f(\lambda_k) = c_k$, $k = 1, 2, \ldots$. An infinite Blaschke product is called an interpolating Blaschke product if its has different zeros and its zeros formed an interpolating sequence.

The Frostman theorem (see for example [1]) say that if $I(z)$ is an inner function in the unit disk, then for almost all $a \in D$ the function

$$\frac{I(z) - a}{1 - \overline{a} I(z)} = b(z)$$

is a Blaschke product. From these theorem it follows that every inner function can be approximated in the unit disk uniformly by Blaschke product (see for ex. [1]).

We will prove the following assertion.

Theorem. *For every $c \in (0, 1)$ there exists a sequence of interpolating Blaschke products $b_m(z) = b_m(z, \lambda_k^{(m)})$, $m \geq 2$, with real positive zeros which converge uniformly in the unit disk to the inner function $I(z) = \exp\left(-\dfrac{1+z}{1-z}\right)$. The zeros $(\lambda_k^{(m)})$of the Blaschke products $b_m(z)$, $m \geq 2$ satisfies the exponential interpolating condition:*

$$\frac{1 - \lambda_{k+1}^{(m)}}{1 - \lambda_k^{(m)}} < c, \quad m \geq 2, \quad k = 1, 2, \ldots \tag{1}$$

The sequence $(\lambda_k^{(m)})$ converge in the hyperbolyc metric in the disk, since $\rho(\lambda_k^{(m+p)}, \lambda_k^{(m)}) \to 0$, $m \to \infty$, independently of k and p.

Proof. Choose $c \in (0, 1)$ and a sequence $\{\rho_k\}_{k=1}^{\infty}$, $0 < \rho_k < 1$, $k = 1, 2, \ldots$, such that

$$\frac{1 - \rho_{k+1}}{1 - \rho_k} \leq \frac{1}{2} c, \quad k = 1, 2, \ldots$$

Then the series

$$\sum_{k=1}^{\infty} (1 - \rho_k) \tag{2}$$

converge (for example we may take $1 - \rho_k = \left(\frac{c}{2}\right)^k$, $c \in (0, 1)$, $k = 1, 2, \ldots$). Define

$$R_k^{(m)} = \frac{1}{m}\Big(1-\frac{1}{m}\Big)^k,$$

$$x_k^{(m)} = \Big(\frac{\rho_k}{1-\sqrt{\rho_k}R_k^{(m)}}\Big)^{(1-\rho_k)^{-3}(R_k^{(m)})^{-1}}, \quad k=1,2,\dots,\ m\ge 2, \qquad (3)$$

$$p_k^{(m)} = \frac{1-x_k^{(m)}}{1+x_k^{(m)}},$$

$$T_k^{(m)} = E[(\exp 2(1-\rho_k)^{-2}\rho_k^{-1}(R_k^{(m)})^{-1})]+1, \qquad (4)$$

$$y_k^{(m)} = \int_0^{p_k^{(m)}} \frac{1-x^{2T_k^{(m)}}}{1-x^2}\,dx, \quad I_k^{(m)} = \int_0^{p_k^{(m)}} \frac{x^{2T_k^{(m)}}}{1-x^2}dx, \qquad (5)$$

$$\lambda_k^{(m)} = \rho_k \exp\big[-(1-\rho_k)^3 R_k^{(m)} y_k^{(m)}\big]$$
$$= (1-\sqrt{\rho_k}R_k^{(m)})\exp(1-\rho_k)^3 R_k^{(m)} I_k^{(m)}, \qquad (6)$$

$k=1,2,\dots$ It is clear that $0<x_k^{(m)}<1$, $0<p_k^{(m)}<1$, $m\ge 2$ $k=1,2,\dots$ Elementary calculations show that

$$x_k^{(m)}T_k^{(m)} > \exp\big[(\rho_k^{-1}(1-\rho_k)^{-2}(R_k^{(m)})^{-1}) + \sqrt{\rho_k}(1-\rho_k)^{-3}\big] \to \infty$$

if $m\to\infty$ independantly of k, and

$$\exp(-2x_k^{(m)}T_k^{(m)}) < \frac{\rho_k(1-\rho_k)^2}{3(1-\rho_k)^2\rho_k R_k^{(m)} + 2 + 2\sqrt{\rho_k}(1-\rho_k)^{-1}\rho_k R_k^{(m)}}$$
$$\le \frac{(1-\rho_k)^2\rho_k R_k^{(m)}}{2} \to 0$$

if $m\to\infty$. Hence we have following estimation for $(1-\rho_k)^2 R_k^{(m)} I_k^{(m)}$:

$$(1-\rho_k)^3 R_k^{(m)} I_k^{(m)} \le (1-\rho_k)^3 R_k^{(m)}\, e^{-2x_k^{(m)}T_k^{(m)}} \log\frac{1}{x_k^{(m)}} \le$$
$$\le \frac{\rho_k(1-\rho_k)^2 R_k^{(m)}}{2}\Big(\frac{1-\rho_k}{\rho_k} - \sqrt{\rho_k}R_k^{(m)}\Big) \to 0, \qquad (7)$$

if $m\to\infty$.

Note that

$$\lim_{m\to\infty}\sum_{k=1}^{\infty} R_k^{(m)} = 1$$

$$\lim_{m\to\infty}\sum_{k=1}^{\infty}(R_k^{(m)})^2 = 0 \tag{8}$$

$$\lim_{m\to\infty}\sum_{k=1}^{\infty}(\sqrt{\rho_k})R_k^{(m)}) = \lim_{m\to\infty}\sum_{k=1}^{\infty}[R_k^{(m)} - (1-\sqrt{\rho_k})R_k^{(m)}] = 1\,,$$

since for every convergent series

$$\sum_{k=1}^{\infty} a_k, a_k > 0, \quad k = 1, 2, \cdots, \ \lim_{m\to\infty}\sum_{k=1}^{\infty} a_k R_k^{(m)} = 0.$$

Using (3), (7) and (8), and the Taylor series expansion of the exponential function, we have

$$\lim_{m\to\infty}\sum_{k=1}^{\infty}(1-\lambda_k^{(m)}) = 1$$

$$\lim_{m\to\infty}\sum_{k=1}^{\infty}(1-\lambda_k^{(m)})^2 = 0. \tag{9}$$

We will proof that the Blaschke products $b_m(z)$, $m \geq 2$, with zero set $\{\lambda_k^{(m)}\}_{k=1}^{\infty}$ converge uniformly in the unit disk to the inner function $I(z)$. First, if $0 < a_k < 1$, $0 < b_k < 1$, $k = 1, 2, \ldots$ and $\prod_{k=1}^{\infty} a_k$ and $\prod_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{\infty} |a_k - b_k|$ converge and

$$\left|\prod_{k=1}^{\infty} a_k - \prod_{k=1}^{\infty} b_k\right| \leq \sum_{k=1}^{\infty} |a_k - b_k|\,. \tag{10}$$

Using the inequality (5) and the inequalyties $|1-\overline{a}b| > 1-|a|$, $|a| < 1$, we have

$$|b_{m+p}(z) - b_m(z)| \leq 2\sum_{k=1}^{\infty}\frac{|\lambda_k^{(m+p)} - \lambda_k^{(m)}|}{(1-\rho_k)^2|\lambda_k^{(m)}||\lambda_k^{(m+p)}|} \leq$$

$$\le 2\sum_{k=1}^{\infty}\frac{\left|\exp\left(-(1-\rho_k)^3R_k^{(m)}y_k^{(m)}\right)-\exp\left(-(1-\rho_k)^3\right)R_k^{(m+p)}y_k^{(m+p)})\right|}{(1-\rho_k)^2(1-m^{-1})^2}$$

$$\le \frac{8(e^2-1)}{\rho_1}\sum_{k=1}^{\infty}(1-\rho_k)^2R_k^{(m)}\to 0. \tag{11}$$

In (11) we use the estimation

$$0<y_k^{(m)}<2(1-\rho_k+(1-\rho_k)^3+\cdots+(1-\rho_k)^{2T_k^{(m)}-1})<\frac{2(1-\rho_k)}{\rho_k}. \tag{12}$$

Hence, by (11), the sequence $(b_m(z))$, $m\ge 2$, of interpolating Blaschke products converge uniformly in the open unit disk. Since

$$\sum_{k=1}^{\infty}R_{k-1}^{(m)}\equiv 1,\quad m\ge 2$$

we have

$$I(z)=\exp\left(-\sum_{k=1}^{\infty}\frac{1+z}{1-z}R_{k-1}^{(m)}\right). \tag{13}$$

In each disk $|z|\le r<1$, we have (using (12))

$$|I(z)-b_m(z)|\le\frac{1}{1-r}\sum_{k=1}^{\infty}(1-\lambda_k^{(m)})^2\left[\exp\left(\frac{2}{1-r}\right)-1\right]\to 0,\quad m\to\infty.$$

Hence $b_m(z)\to I(z)$ in the unit disk. According to (11) the convergence is uniform. But we ask how to estimate $|I(z)-b_m(z)|$ in the unit disk up by a positive function which depends only of m and converge to 0 if $m\to\infty$,

We proceed as follows. The function

$$h(z)=\frac{1}{|1-z|^{2\alpha}}\exp\left(-\frac{1-|z|^2}{|1-z|^2}\right),\quad \alpha\ge 1 \tag{14}$$

is bounded in the unit disk. To proof this, we use the polar representation

$$\begin{aligned}1-x&=r\cos\theta\\ y&=r\sin\theta\end{aligned} \tag{15}$$

since $z=x+iy\in D$, we have $1-|z|^2=r\cos\theta-r^2>0$ and $|1-z|^2=r^2$, the function (14) is transformated in the following way

$$h(z)=h(\theta,r)=\frac{1}{r^{2\alpha-1}}\exp\left(-\frac{2\cos\theta-r}{r}\right) \tag{16}$$

which is clear that it is bounded e.t. there exist $0 < M < 1$ such that

$$0 < h(z) < M < 1. \tag{17}$$

Define

$$I_m(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{1+z}{1-z}(1-\lambda_k^{(m)})\right). \tag{18}$$

It is clear that $I_m(z)$, $z \in D$, are inner functions for every $m \geq 2$. Since

$$\lim_{m\to\infty} \sum_{k=1}^{\infty}(1-\lambda_k^{(m)}) = 1, \tag{19}$$

and the function $h(z)$ is bounded by 1 in the unit disk,using series expansion of the exponential function, elementary calculations shows that

$$\begin{aligned} |I_m(z) - I(z)| &= |I(z)||1-(I(z))^{-1}I_m(z)| \leq \\ &\leq \sum_{k=1}^{\infty}\left[(R_{k-1}^{(m)} - (1-\lambda_k^{(m)}) + \frac{2}{2!}(R_{k-1}^{(m)} - (1-\lambda_k^{(m)}))^2 + \cdots\right] \\ &< \sum_{k=1}^{\infty}\left[(R_{k-1}^{(m)} - (1-\lambda_k^{(m)}))\right] + \\ &+ \sum_{k=1}^{\infty}\left[(R_{k-1}^{(m)} - (1-\lambda_k^{(m)}))\right]^2 \left(\frac{2^2}{2!} + \frac{2^3}{3!} + \cdots\right) \to 0 \end{aligned} \tag{20}$$

if $m \to \infty$, and so

$$I(z) = \lim_{m\to\infty} I_m(z)$$

uniformly in the unit disk. Using the inequality

$$\frac{1}{|1-\overline{a}z|} < \frac{2}{|e^{i\theta}-z|}, \quad \left|\frac{a-z}{1-\overline{a}z}\right| < 1 \quad a, z \in D,\ \theta = \arg a \tag{21}$$

the Taylor expansion of the exponential function, estimation (17) of the function $h(z)$, we have

$$\begin{aligned} |I_m(z) - b_m(z)| &= |I_m(z)||1-(I_m(z))^{-1}b_m(z)| \leq \\ &\leq 2(e-1)\sum_{k=1}^{\infty}(1-\lambda_k^{(m)})^2 \to 0, \quad m \to \infty \end{aligned}$$

uniformly in the whole open unit disk.

For the zeros $\lambda_k^{(m)}$, $m \geq 2$, $k = 1, 2, \ldots$ of Blaschke products $b_m(z)$, $m \geq 2$, we have

$$\frac{1-\lambda_{k+1}^{(m)}}{1-\lambda_k^{(m)}} = \frac{1-\rho_{k+1}\exp\left[-(1-\rho_{k+1})^3 R_{k+1}^{(m)} y_{k+1}^{(m)}\right]}{1-\rho_k\exp\left[-(1-\rho_k)^3 R_k^{(m)} y_k^{(m)}\right]} \leq$$
$$\leq \frac{1-\rho_{k+1}}{1-\rho_k} + \frac{\rho_{k+1}(1-\rho_{k+1})^3 R_{k+1}^{(m)}(1-\rho_{k+1})\rho_{k+1}^{-1}}{1-\rho_k} < c\,. \tag{22}$$

In a simillar way can be proved that for the inner function

$$I(z) = \exp\left(-\frac{e^{i\theta}+z}{e^{i\theta}-z}\right), \quad a > 0 \tag{23}$$

for every $c \in (0,1)$ there exists a sequence of interpolating Blaschke products $b_m(z, \lambda_k^{(m)})$, $m \geq 2$, withzeros $\lambda_k^{(m)}$, $\arg\lambda_k^{(m)} = \theta_k$, $m \geq 2$, $k = 1, 2, \ldots$ which coverge uniformly in unit disk to the inner function $I(z)$. The zeros $\lambda_k^{(m)}$ can be defined in the following way

$$\lambda_k^{(m)} = \rho_k \exp\left[-(1-\rho_k)^3 a R_k^{(m)} y_k^{(m)} + i\theta\right], \quad m \geq 2, \quad k = 1, 2, \ldots \tag{24}$$

The zeros (24) satisfy the exponential interpolating property

$$\frac{1-|\lambda_{k+1}^{(m)}|}{1-|\lambda_k^{(m)}|} < c, \quad m \geq 2, \quad k = 1, 2, \ldots$$

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ЗА УНИФОРМНАТА АПРОКСИМАЦИЈА НА ЕДНА ВНАТРЕШНА ФУНКЦИЈА

Никола Пандески

Резиме

Внатрешната функција $\exp\left(-\frac{1+z}{1-z}\right)$ е апроксимација во единичниот диск рамномерно со интерпретациони Блашкеови производи.

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