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INTERPOLATION IN THE CLASS M^p, $0 < p < 1$

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Abstract

In this paper we give sufficient conditions such that the interpolation problems for the sequences $(\lambda_k)_{k=1}^{\infty}$ and $(c_k)_{k=1}^{\infty}$, has a solution in the class M^p, $0 < p < 1$.

Let D be the unit disk in the complex plane, f holomorphic function in the disk D,

$$Mf(\theta) = \sup_{0 \le r < 1} |f(re^{i\theta})|$$

and

$$\log^+ a = \begin{cases} 0 & 0 < a < 1 \\ \log a & a \ge 1 \end{cases}$$

We denote by M^p the class of holomorphic functions in D such that

$$\int_0^{2\pi} \left(\log^+ Mf(\theta)\right)^p d\theta < +\infty.$$

For $0 < p < 1$ the class M^p contain the Nevalina class N. Gavrilov and Subotin [4], proved that the functions in the class M^p, $0 < p < 1$ has boundary values.

Let $(\lambda_k)_{k=1}^{\infty}$ be a given sequence in the unit disk D such that

$$\sum_{k=1}^{\infty}(1-|\lambda_k|)^{\alpha+1} < +\infty, \quad \alpha > -1. \tag{1}$$

We look for a function $f \in M^p,\ 0 < p < 1$, such that

$$f(\lambda_k) = c_k, \quad k = 1, 2, \ldots$$

where $(c_k)_{k=1}^{\infty}$ is a given sequence.

Let $b_p(z)$ be the Blaschke-Naftalevich-Djarbashjan product in the with disk D

$$b_p(z) = \prod_{j=1}^{\infty}\left(1 - \frac{1-|\lambda_j|^2}{1-\overline{\lambda}_j z}\right)\exp\left\{\sum_{j=1}^{p}\frac{1}{j}\left(1-\frac{1-|\lambda_j|^2}{1-\overline{\lambda}_j z}\right)^j\right\}, \quad p = [\alpha].$$

Note that $b_p(z)$ converge if (1) is satisfied and $|b_p(z)| < 1$ for $z \in D$. If $-1 < \alpha < 1$, we have the usual Blaschke product.

Let

$$b_{p,k}(z) = \prod_{j, j\neq k}\left(1 - \frac{1-|\lambda_j|^2}{1-\overline{\lambda}_j\, z}\right)\exp\left\{\sum_{j=1}^{p}\frac{1}{j}\left(\frac{1-|\lambda_j|}{1-\overline{\lambda}_j\, z}\right)^j\right\}$$

Theorem. *Suppose that*

$$\sum_{k=1}^{\infty}(1-|\lambda_k|)^{\alpha+1} < +\infty \quad \text{and} \quad \arg\lambda_k = \theta_0\,.$$

Let $(c_k)_{k=1}^{\infty}$ be a sequence such that

$$\sum_{k=1}^{\infty}\frac{|c_k|}{|b_{p,k}(\lambda_k)|} < \infty$$

Then there exist $f \in M^p$, $0 < p < 1$, such that $f \notin N$ and $f(\lambda_k) = c_k$, $k = 1, 2, \ldots$

Proof. Let $0 < p < 1$. Define

$$f(z)=\sum_{k=1}^{\infty}c_k\frac{b_{p,k}(z)}{b_{p,k}(\lambda_k)}\times$$
$$\times\exp\Big(\sum_{k=1}^{\infty}\frac{(1-|\lambda_k|)^{\frac{\alpha+1}{p}+\frac{\beta}{p}}}{(1-e^{i\theta_0}z)^{\beta}}-\sum_{k=1}^{\infty}(1-|\lambda_k|)^{\frac{\alpha+1}{p}+\frac{\beta}{p}}-\beta\Big).$$

Then $f\in M^p$ for every $0<p<1$, $f\notin N$ and $f(\lambda_k)=c_k$, $k=1,2,\ldots$, for some $\beta>1$.

It is clear that f is a holomorphic function.

To prove that $f\notin N$ and $f\in M^p$, $0<p<1$ it is enough to prove that the function

$$g(z)=\exp\Big(\sum_{k=1}^{\infty}\frac{(1-|\lambda_k|)^{\frac{\alpha+1}{p}+\frac{\beta}{p}}}{(1-e^{-i\theta_0}z)^{\beta}}\Big),$$

for some β it is in M^p, $0<p<1$, but it is not in N.

We have for $z=re^{i\theta}\in D$

$$|g(re^{i\theta})|=\exp\sum_{k=1}^{\infty}d_k\frac{1-\beta r\cos(\theta-\theta_0)+\cdots}{(1-2r\cos(\theta-\theta_0)+r^2)^{\beta}}$$
$$=g_0(re^{i\theta})g_1(re^{i\theta}),$$

where

$$g_0(re^{i\theta})=\exp\left(\sum_{k=1}^{\infty}\frac{d_k}{(1-2r\cos(\theta-\theta_0)+r^2)^{\beta}}\right),$$

and $d_k=(1-|\lambda_k|)^{\frac{\alpha+1}{p}+\frac{\beta}{p}}$. For $g_0(re^{i\theta})$ we have

$$\sup_{0\le r<1}\int_0^{2\pi}\log^+|g_0(re^{i\theta})|\,d\theta=+\infty$$

and so $g_0\notin N$, which implies that $g\notin N$ and $f\notin N$.

For $g_0(re^{i\theta})$ we have

$$Mg_0(re^{i\theta})\le\exp\left(\sum_{k=1}^{\infty}\frac{d_k}{|\sin(\theta-\theta_0)|^{2\beta}}\right).$$

and

$$\int_0^{2\pi}(\log^+ Mg_0(re^{i\theta}))^p d\theta \le \int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{d_k}{|\sin(\theta-\theta_0)|^{2\beta}}\right)^p d\theta$$

$$= \left(\sum_{k=1}^{\infty} d_k\right)^p \int_0^{2\pi}\frac{d\theta}{|\sin\theta|^{2\beta p}}.$$

We take $2\beta p = \frac{1}{2}$. So $\beta = \frac{1}{4p} > 1$ if $0 < p < \frac{1}{4}$.

Then

$$\int_0^{2\pi}\frac{d\theta}{|\sin\theta|^{2\beta p}} = \int_0^{2\pi}\frac{d\theta}{|\sin\theta|^{\frac{1}{2}}} < +\infty,$$

and because

$$\left(\sum_{k=1}^{\infty} d_k\right)^p < +\infty,$$

we have

$$\int_0^{2\pi}\log^+(Mg_0(re^{i\theta}))^p d\theta < \infty \quad \text{for} \quad 0 < p < \frac{1}{4}.$$

Because $M^p \subset M^q$ for $0 < q < p$ we have $g_0 \in M^p$ for $0 < p < 1$.

Other conditions

Let $0 < p < 1$ and

$$\sum_{k=1}^{\infty}(1-|\lambda_k|)^p < \infty.$$

If $\arg\lambda_k = \theta_0$, and

$$\sum_{k=1}^{\infty}\frac{(1-|\lambda_k|)^p|\log c_k|^p}{|b_k(\lambda_k)|} < \infty$$

then

$$g(z) = \exp\sum_{k=1}^{\infty}\frac{1-|\lambda_k|}{1-e^{-i\theta_0 z}}\;\frac{b_k(z)}{b_k(\lambda_k)}\;\log c_k$$

belongs to M^p, $0 < p < 1$ and

$$g(\lambda_k) = c_k, \quad k = 1, 2, \dots$$

Here $b(z)$ is a usal Blashke product and

$$b_k(z) = \prod_{j \neq k} \frac{\lambda_j - z}{1 - \overline{\lambda}_j z} \frac{|\lambda_j|}{\lambda_j}.$$

The prove is simillar.

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ИНТЕРПОЛАЦИЈА ВО КЛАСАТА M^p, $0 < p < 1$

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Р е з и м е

Во трудот се даваат доволни услови така да интерполационот проблем за низите $(\lambda_k)_{k=1}^{\infty}$ и $(c_k)_{k=1}^{\infty}$, има решение во класата M^p, $0 < p < 1$.

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