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# ON THE ORIENTABILITY OF THE GRASSMANN MANIFOLDS

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## Abstract

In [3] it is proved that the Grassmann manifold $G_{p,q}(R)$ is orientabled if and only if p+q is even number. In this paper it is given alternative proof of this statement.

First we give some preliminaries. Let $p$ and $q$ are positive integers and $n = p + q$. The elements of the Grassmann manifold $G_{p,q}(R)$ are $p$–dimensional subspaces of the Euclidean space $R^n$ and the elements of the oriented Grassmann manifold $G^+_{p,q}(R)$ are oriented $p$–dimensional subspaces of the Euclidean space $R^n$. Let

$$\mathbf{x}_{(r)} = (x^1_r, x^2_r, \cdots, x^n_r) \qquad r = 1, \cdots, p$$

be $p$ linearly independent vectors which generate a given subspace $V_p$ of $R^n$, and let

$$\mathbf{y}_{(s)} = (y^1_s, y^2_s, \cdots, y^n_s) \qquad s = 1, \cdots, q$$

be linearly independent vectors orthogonal to $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(p)}$, i.e. they generate the normal space $N_q$ of $V_p$. For $\alpha_1, \cdots, \alpha_p \in \{1, \cdots, n\}$ we denote

$$M_{\alpha_1 \cdots \alpha_p} = \begin{bmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_p} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_p} \\ \cdots & \cdots & \cdots & \cdots \\ x_p^{\alpha_1} & x_p^{\alpha_2} & \cdots & x_p^{\alpha_p} \end{bmatrix}, \qquad \text{and} \qquad \Delta_{\alpha_1 \cdots \alpha_p} = \det M_{\alpha_1 \cdots \alpha_p}.$$

The linearly independence of the vectors $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(p)}$, means that there exist $p$ different numbers $\alpha_1, \cdots, \alpha_p$ such that $\Delta_{\alpha_1 \cdots \alpha_p} \neq 0$. Then using linear combination of the vectors $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(p)}$ we can obtain new vectors $\mathbf{x}'_{(1)}, \cdots, \mathbf{x}'_{(p)}$, such that $M' = I_{p \times p}$, i.e. $M'$ is the unit matrix. Then the rest $pq$ elements $x'^i_j$ for $1 \leq j \leq p$ and $i \in \{1, \cdots, n\} - \{\alpha_1, \cdots, \alpha_p\}$ can be considered as coordinates of $G_{p,q}(R)$ in respect to the coordinate neighborhood $U_{\alpha_1 \cdots \alpha_p}$. Moreover we can assume that $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq n$. Hence $G_{p,q}(R)$ is a differentiable manifold covered by $\binom{n}{p}$ charts. In order $G^+_{p,q}(R)$ to become a differentiable manifold, for each sequence $1 \leq \alpha_1 \leq \cdots \leq \alpha_p \leq n$ we should consider two neighborhoods $U^+_{\alpha_1 \cdots \alpha_p}$ and $U^-_{\alpha_1 \cdots \alpha_p}$ corresponding to $\Delta_{\alpha_1 \cdots \alpha_p} > 0$ and $\Delta_{\alpha_1 \cdots \alpha_p} < 0$ respectively. Indeed these two neighborhoods are nonintersecting on the manifold of all $p \times n$–matrices of rank $p$ but they coincide to $U_{\alpha_1 \cdots \alpha_p}$ on the Grassmann manifold $G_{p,q}(R)$. The following lemma gives the relation between the coordinates $x'^i_j$ and $x^i_j$.

**Lemma 1.** *If* $\Delta_{\alpha_1 \cdots \alpha_p} \neq 0$, *then*

$$x'^i_1 = \frac{\Delta_{i \alpha_2 \cdots \alpha_p}}{\Delta_{\alpha_1 \cdots \alpha_p}}, \quad x'^i_2 = \frac{\Delta_{\alpha_1 i \alpha_3 \cdots \alpha_p}}{\Delta_{\alpha_1 \cdots \alpha_p}}, \quad \cdots, \quad x'^i_p = \frac{\Delta_{\alpha_1 \cdots \alpha_{p-1} i}}{\Delta_{\alpha_1 \cdots \alpha_p}}, \qquad (1)$$

$$i = 1, \cdots, n.$$

**Proof.** It is obvious that $x'^{\alpha_r}_r = 1$, and $x'^{\alpha_s}_r = 0$ for $s \neq r$, i.e. $x'^{\alpha_s}_r = \delta^s_r$. Let $i \neq \alpha_1, \cdots, \alpha_p$ and $x'^i_1, \cdots, x'^i_p$ be the solution of the following system

$$M^{-1}_{\alpha_1 \cdots \alpha_p} \begin{bmatrix} x^i_1 \\ \cdot \\ \cdot \\ \cdot \\ x^i_p \end{bmatrix} = \begin{bmatrix} x'^i_1 \\ \cdot \\ \cdot \\ \cdot \\ x'^i_p \end{bmatrix}, \qquad \text{i.e.} \qquad M_{\alpha_1 \cdots \alpha_p} \begin{bmatrix} x'^i_1 \\ \cdot \\ \cdot \\ \cdot \\ x'^i_p \end{bmatrix} = \begin{bmatrix} x^i_1 \\ \cdot \\ \cdot \\ \cdot \\ x^i_p \end{bmatrix}.$$

Hence, using the Cramer formulas we obtain (1). □

Note that the vectors $\mathbf{x}'_{(1)}, \cdots, \mathbf{x}'_{(p)}$ do not depend on the choice of the linearly independent vectors $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(p)}$ but depend only on the subspace $V_p$ generated by $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(p)}$. The coordinates $(x'^i_j)$ are analogous to the homogeneous coordinates for projective spaces. Now we will see the relation between these coordinates and the coordinates $y^i_j$.

Let $\beta_1, \cdots, \beta_q$ be the complementary sequence in increasing order of $\{\alpha_1, \cdots, \alpha_p\}$ up to $\{1, 2, \cdots, n\}$. For arbitrary $\gamma_1, \cdots, \gamma_q$ we define

$$N_{\gamma_1\cdots\gamma_q} = \begin{bmatrix} y_1^{\gamma_1} & y_1^{\gamma_2} & \cdots & y_1^{\gamma_q} \\ y_2^{\gamma_1} & y_2^{\gamma_2} & \cdots & y_2^{\gamma_q} \\ \cdots & \cdots & \cdots & \cdots \\ y_q^{\gamma_1} & y_q^{\gamma_2} & \cdots & y_q^{\gamma_q} \end{bmatrix}, \qquad \text{and} \qquad D_{\gamma_1\cdots\gamma_q} = \det N_{\gamma_1\cdots\gamma_q}.$$

**Lemma 2.** *Under the previous notations, if* $D_{\beta_1\cdots\beta_q} \neq 0$, *then*

$$x_r'^{\beta_s} = (-1)^s \cdot D_{\alpha_r\beta_1\cdots\beta_{s-1}\beta_{s+1}\cdots\beta_q} / D_{\beta_1\cdots\beta_q} \quad r = 1, \cdots, p, s = 1, \cdots, q \quad (2)$$

*and* $x_r'^{\alpha_s} = \delta_r^s$.

**Proof.** We should verify that the vectors $\mathbf{x}'_{(1)}, \cdots, \mathbf{x}'_{(p)}$ satisfy $\mathbf{x}'_{(r)} \cdot \mathbf{y}_{(s)} = 0$ for $1 \leq r \leq p, 1 \leq s \leq q$ and $x_r'^{\alpha_s} = \delta_r^s$ which is obvious.

Let $1 \leq r \leq p, 1 \leq s \leq q$. Then

$$\mathbf{x}'_{(r)} \cdot \mathbf{y}_{(s)} = \sum_{a=1}^{n} x_r'^a \cdot y_s^a = x_r^{\alpha_r} \cdot y_s^{\alpha_r} + \sum_{t=1}^{q} x_r'^{\beta_t} \cdot y_s^{\beta_t} =$$

$$= y_s^{\alpha_r} + \sum_{t=1}^{q} (-1)^t \cdot y_s^{\beta_t} D_{\alpha_r\beta_1\cdots\beta_{t-1}\beta_{t+1}\cdots\beta_q} / D_{\beta_1\cdots\beta_q} =$$

$$= \frac{1}{D_{\beta_1\cdots\beta_q}} \left[ y_s^{\alpha_r} \cdot D_{\beta_1\cdots\beta_q} + \sum_{t=1}^{q} (-1)^t \cdot y_s^{\beta_t} D_{\alpha_r\beta_1\cdots\beta_{t-1}\beta_{t+1}\cdots\beta_q} \right] =$$

$$= \frac{1}{D_{\beta_1\cdots\beta_q}} \cdot \begin{vmatrix} y_s^{\alpha_r} & y_s^{\beta_1} & y_s^{\beta_2} & \cdots & y_s^{\beta_q} \\ y_1^{\alpha_r} & y_1^{\beta_1} & y_1^{\beta_2} & \cdots & y_1^{\beta_q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_q^{\alpha_r} & y_q^{\beta_1} & y_q^{\beta_2} & \cdots & y_q^{\beta_q} \end{vmatrix} = 0. \quad \square$$

Now we are going to find the Jacobian between the two coordinate neighborhoods $U_{\alpha_1\cdots\alpha_p}$ and $U_{\gamma_1\cdots\gamma_p}$. Thus we introduce the notations

$$u^i_{r(\alpha_1\cdots\alpha_p)} = \frac{\Delta_{\alpha_1\cdots\alpha_{r-1}i\alpha_{r+1}\cdots\alpha_p}}{\Delta_{\alpha_1\cdots\alpha_p}}$$

for $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq n, 1 \leq i \leq n, 1 \leq r \leq p$, such that for $i \neq \alpha_1, \cdots, \alpha_p$ we obtain the coordinates of the point determined by

$\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(p)}$, with respect to the coordinate neighborhood $U_{\alpha_1\cdots\alpha_p}$. Analogously we introduce the notations

$$u^j_{s(\gamma_1\cdots\gamma_p)} = \frac{\Delta_{\gamma_1\cdots\gamma_{s-1}j\gamma_{s+1}\cdots\gamma_p}}{\Delta_{\gamma_1\cdots\gamma_p}}$$

for $1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq n, 1 \leq j \leq n, 1 \leq s \leq p$. Our aim is to find the Jacobi matrix

$$J = \left[\frac{\partial u^i_{r(\alpha_1\cdots\alpha_p)}}{\partial u^j_{s(\gamma_1\cdots\gamma_p)}}\right]$$

which is $pn \times pn$ matrix. By neglecting the elements for $i = \alpha_1, \cdots, i = \alpha_p, r = 1, \cdots, p$ and the elements for $j = \gamma_1, \cdots, j = \gamma_p$ we get the required Jacobi matrix $\left[J^{(\alpha_1\cdots\alpha_p)}_{(\gamma_1\cdots\gamma_p)}\right]$ with the required determinant.

First we consider the special case $\gamma_i = \alpha_i$ for $i \neq t$ and $\gamma_t = \alpha_t + 1$. According to the definition of the coordinates $u^i_{r(\alpha_1\cdots\alpha_p)}$ and $u^j_{s(\gamma_1\cdots\gamma_p)}$ it follows that

$$r \neq t: \qquad u^i_{r(\alpha_1\cdots\alpha_p)} = u^i_{r(\gamma_1\cdots\gamma_p)} - u^i_{t(\gamma_1\cdots\gamma_p)} \cdot \frac{u^{\alpha_t}_{r(\gamma_1\cdots\gamma_p)}}{u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}}$$

$$r = t: \qquad u^i_{t(\alpha_1\cdots\alpha_p)} = \frac{u^i_{t(\gamma_1\cdots\gamma_p)}}{u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}}.$$

Hence assuming that $i \neq \alpha_1, \cdots, \alpha_p$ for each $r$ and $j \neq \gamma_1, \cdots, \gamma_p$ for each $s$, we obtain

$$\begin{aligned} r \neq t: \qquad \frac{\partial u^i_{r(\alpha_1\cdots\alpha_p)}}{\partial u^j_{s(\gamma_1\cdots\gamma_p)}} =& \delta^i_j \delta^r_s - \delta^i_j \delta^t_s \cdot \frac{u^{\alpha_t}_{r(\gamma_1\cdots\gamma_p)}}{u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}} - \\ & - \delta^{\alpha_t}_j \delta^r_s \cdot \frac{u^i_{t(\gamma_1\cdots\gamma_p)}}{u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}} + \\ & + \delta^{\alpha_t}_j \delta^t_s \cdot u^i_{t(\gamma_1\cdots\gamma_p)} \cdot \frac{u^{\alpha_t}_{r(\gamma_1\cdots\gamma_p)}}{\left(u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}\right)^2}, \end{aligned} \tag{3a}$$

$$\begin{aligned} r = t: \qquad \frac{\partial u^i_{t(\alpha_1\cdots\alpha_p)}}{\partial u^j_{s(\gamma_1\cdots\gamma_p)}} =& \delta^i_j \delta^t_s \cdot \frac{1}{u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}} - \\ & - \delta^{\alpha_t}_j \delta^t_s \cdot u^i_{t(\gamma_1\cdots\gamma_p)} \cdot \frac{1}{\left(u^{\alpha_t}_{t(\gamma_1\cdots\gamma_p)}\right)^2}. \end{aligned} \tag{3b}$$

From any given $p \times n$ matrix $[A_r^i], (i = 1, \cdots, n; r = 1, \cdots, p)$ we construct $pn$–vector with coordinates

$$(A_1^1, A_1^2, \cdots, A_1^n, A_2^1, A_2^2, \cdots A_2^n, \cdots, A_p^1, A_p^2, \cdots, A_p^n).$$

Thus from (3) we form the Jacobi $pn \times pn$ matrix and then we reduce it to the required $pq \times pq$ matrix. This $pq \times pq$ matrix $[\partial u^i_{r(\alpha_1 \cdots \alpha_p)} / \partial u^j_{s(\gamma_1 \cdots \gamma_p)}]$ can be partitioned into

$$[(p-1)+(p-1)(q-1)+1+(q-1)] \times [(p-1)+(p-1)(q-1)+1+(q-1)]$$

block matrices as follows

$$\begin{bmatrix} B & 0 & * & 0 \\ * & I & * & * \\ 0 & 0 & E & 0 \\ 0 & 0 & * & D \end{bmatrix}$$

where

$$B = -I \cdot \frac{u^{\alpha_t+1}_{t(\gamma_1 \cdots \gamma_p)}}{u^{\alpha_t \cdot}_{t(\gamma_1 \cdots \gamma_p)}}, \quad E = -\frac{1}{\left(u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)}\right)^2} \cdot \left[u^{\alpha_t+1}_{t(\gamma_1 \cdots \gamma_p)}\right], \quad D = I \cdot \frac{1}{u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)}}$$

and $I$ denotes the unit matrix of the corresponding order. Hence for the considered $p$–typles $(\alpha_1, \cdots, \alpha_p)$ and $(\gamma_1, \cdots, \gamma_p)$ we obtain

$$J^{(\alpha_1 \cdots \alpha_p)}_{(\gamma_1 \cdots \gamma_p)} = \det B \cdot E \cdot \det D =$$

$$= (-1)^{p-1} \cdot \left(\frac{u^{\alpha_t+1}_{t(\gamma_1 \cdots \gamma_p)}}{u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)}}\right)^{p-1} \cdot \frac{-u^{\alpha_t+1}_{t(\gamma_1 \cdots \gamma_p)}}{\left(u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)}\right)^2} \cdot \frac{1}{\left(u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)}\right)^{q-1}} =$$

$$= (-1)^p \cdot \frac{\left(u^{\alpha_t+1}_{t(\gamma_1 \cdots \gamma_p)}\right)^p}{\left(u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)}\right)^n}.$$

According to the definition we get

$$u^{\alpha_t+1}_{t(\gamma_1 \cdots \gamma_p)} = \frac{\Delta_{\gamma_1 \cdots \gamma_p}}{\Delta_{\gamma_1 \cdots \gamma_p}} = 1 \qquad \text{and} \qquad u^{\alpha_t}_{t(\gamma_1 \cdots \gamma_p)} = \frac{\Delta_{\alpha_1 \cdots \alpha_p}}{\Delta_{\gamma_1 \cdots \gamma_p}},$$

and hence

$$J^{(\alpha_1\cdots\alpha_p)}_{(\gamma_1\cdots\gamma_p)} = (-1)^p \cdot \left(\frac{\Delta_{\gamma_1\cdots\gamma_p}}{\Delta_{\alpha_1\cdots\alpha_p}}\right)^n.$$

We obtained this formula for the special case $\gamma_i = \alpha_i$ for $i \neq t$ and $\gamma_t = \alpha_t + 1$. Applying the previous formula more times, for the general case we obtain the following

**Proposition 3.** *The Jacobian between two coordinate neighborhoods $U_{\alpha_1\cdots\alpha_p}$ and $U_{\gamma_1\cdots\gamma_p}$ is*

$$J^{(\alpha_1\cdots\alpha_p)}_{(\gamma_1\cdots\gamma_p)} = (-1)^{p\cdot(\alpha_1+\cdots+\alpha_p+\gamma_1+\cdots+\gamma_p)} \cdot \left(\frac{\Delta_{\gamma_1\cdots\gamma_p}}{\Delta_{\alpha_1\cdots\alpha_p}}\right)^n. \ \square \tag{4}$$

According to (4) we are able to answer whether the Grassmann manifold $G_{p,q}(R)$ is orientabled or not. Note that one manifold covered with coordinate neighborhoods is orientabled if and only if some of the coordinate neighborhoods change their orientations (for example the first coordinate is multiplied by -1) such that all Jacobians are positive. Note also that the manifold $G^+_{p,q}(R)$ is covered by the previously $2\binom{n}{p}$ coordinate neighborhoods $U^+_{\alpha_1\cdots\alpha_p}$ and $U^-_{\alpha_1\cdots\alpha_p}$. In the Grassmann manifold the coordinates are $\Delta_{\gamma_1\cdots\gamma_p}/\Delta_{\alpha_1\cdots\alpha_p}$, and thus they should be invariant if all determinants $\Delta_{\alpha_1\cdots\alpha_p}$ change their signs. It is sufficient to consider the case when two vectors $\mathbf{x}_{(i)}$ and $\mathbf{x}_{(j)}$ change their places. Thus $G^+_{p,q}(R)$ twice covers the Grassmann manifold $G_{p,q}(R)$. In [1] it has been proven that

$$G_{p,q}(R) = O(p+q)/O(p) \times O(q)$$

and

$$G^+_{p,q}(R) = SO(p+q)/SO(p) \times SO(q),$$

where $O(r)$ denotes the group of orthogonal real matrices of order $r$ and $SO(r)$ denotes the group of orthogonal real matrices of order $r$ with determinant equal to 1. Hence $G_{p,q}(R)$ is orientabled if some coordinate neighborhoods $U_{\alpha_1\cdots\alpha_p}$ change their signs which means that simultaneously should be changed the signs in $U^+_{\alpha_1\cdots\alpha_p}$ and $U^-_{\alpha_1\cdots\alpha_p}$, such that all Jacobians are positive. Now we will examine the orientability of $G_{p,q}(R)$. Two cases are possible.

i) $n$ is even number. If $p$ is also even, then obviously $G_{p,q}(R)$ is orientabled manifold. So assume that $p$ and $q$ are odd numbers. Then in the coordinate neighborhoods $U^+_{\alpha_1\cdots\alpha_p}$ and $U^-_{\alpha_1\cdots\alpha_p}$ simultaneously we change the sign if and only if $\alpha_1 + \cdots + \alpha_p$ is odd number. Hence all Jacobians become positive and $G_{p,q}(R)$ is orientabled manifold.

ii) $n$ is odd number. It is easy to see that in order to orientate $G^+_{p,q}(R)$ it has to appear only one of the cases:

– All $U^{+}_{\alpha_1\cdots\alpha_p}$ such that $\alpha_1+\cdots+\alpha_p$ is even number and all $U^{-}_{\alpha_1\cdots\alpha_p}$ such that $\alpha_1+\cdots+\alpha_p$ is odd number, should change the signs,

– All $U^{+}_{\alpha_1\cdots\alpha_p}$ such that $\alpha_1+\cdots+\alpha_p$ is odd number and all $U^{-}_{\alpha_1\cdots\alpha_p}$ such that $\alpha_1+\cdots+\alpha_p$ is even number, should change the signs.

Hence it is clear that $G^{+}_{p,q}(R)$ is orientabled manifold and $G_{p,q}(R)$ is not orientabled manifold in this case. So, we have proven the following

**Proposition 4.** *The manifold $G_{p,q}(R)$ is orientabled if and only if $n$ is even number .* □

## References

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# ЗА ОРИЕНТАБИЛНОСТА НА ГРАСМАНОВИТЕ МНОГУОБРАЗИЈА

Костадин Тренчевски

**Р е з и м е**

Во [3] е докажано дека Грасмановото многуобразие $G_{p,q}(R)$ е ориентабилно ако и само ако $p+q$ е парен број. Во овој труд е даден алтернативен доказ на ова тврдење.

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