Математички Билтен
26 (LII)
2002 (43–46)
Скопје, Македонија

ISSN 0351-336X

APPROXIMATION OF A FINITE BLASCHKE PRODUCT BY INFINITE ONE

Nikola Pandeski

Abstract

Let $b_n(z)$ be a finite Blaschke product in the unit disk in the plane. We find a sequence infinite Blaschke product which converge to $b_n(z)$ uniformly on the compact subsets of the unit disk.

Let U be the unit disk in the plane. We call the function $I(z)$, $z \in U$, an inner function if its radial boundury function $I^*(e^{i\theta})$ satisfy $|I^*(e^{i\theta})| = 1$ $a.e$ on ∂U. A infinite Blaschke product in the unit disk is a function

$$b(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{1 - \overline{\lambda}_k} \frac{|\lambda_k|}{\lambda_k}, \quad \sum_{k=1}^{\infty}(1 - |\lambda_k|) < \infty . \tag{1}$$

It is clear that the numbers λ_k, $k \in N$, are zeros of the Blaschke product $b(z)$. A finite Blaschke product is a Blaschke product with finite many zeros $\lambda_1, \lambda_2, \ldots, \lambda_k$:

$$b_n(z) = \prod_{k=1}^{n} \frac{\lambda_k - z}{1 - \overline{\lambda}_k} \frac{|\lambda_k|}{\lambda_k} . \tag{2}$$

By Frostman theorem, (see for ex. [1]) if $I(z)$ is an inner function, than for almost all $a \in U$ the function

$$\frac{I(z) - a}{1 - \overline{a}\, I(z)} = b(z) , \quad z \in U \tag{3}$$

is a Blaschke product.

Theorem. *Let $b_n(z)$ is a finite Blaschke product. Then there exist a sequence of infinite Blaschke products $b_m(z) = b_m(\lambda_k^{(m)}, z)$, $m \in N$ which converge to $b_n(z)$ uniformly on the compact subsets of the unit disk.*

Proof. For every $m \in N$, define

$$u_m(z) = \exp\left(-\frac{1+z}{1-z}\frac{1}{m}\right), \quad z \in U. \tag{4}$$

It easy to chek that $u_m(z) \to 1$, $z \in U$, uniformly on the compact subsets of the unit disk.

Let $h_m(z) = b_n(z)u_m(z)$, $z \in U$, $m \in N$. The functions $h_m(z)$, $m \in N$ are inner functions in the unit disk. We choose a sequence $(a_m)_{m=1}^{\infty}$ in the unit disk such that $a_m \to 0$, $m \to \infty$ and

$$\frac{h_m(z) - a_m}{1 - \overline{a}_m h_m(z)} = B_m(z), \quad m \in N, \tag{5}$$

are Blaschke products. Since the equation

$$h_m(z) = a_m \tag{6}$$

has infinite different zeros zeros, the Blaschke product in (5) is an infinitive Blaschke product. If $m - \infty$, then

$$B_m(z) \to b_n(z) \tag{7}$$

uniformly on the compact subsets of the unit disk.

To find the zeros of the equation (6) is not easy problem. Another way to solve this problem is the folowing. For $m \geq 2$, $x > 0$, define

$$r_k^{(m)} = \frac{1}{m}\left(1 - \frac{1}{xm}\right)^k, \quad k = 1, 2, \ldots \tag{8}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the zeros of the finite Blaschke product. We may suppose that $|\lambda_1| < |\lambda_2|, < \cdots < |\lambda_n|$. Put $\lambda_{n+1} = \lambda_1$, $\lambda_{n+2} = \lambda_2, \ldots, \lambda_{n+k} = \lambda_k$, $k = 1, 2, \ldots$ Let

$$A_m = \sum_{n+1}^{\infty}\left[(1 - |\lambda_k|)^{1/k} - \frac{1}{\sqrt{m}}\right]^k, \quad m \geq 2, \quad m \geq 2 \tag{9}$$

$$|\lambda_k^{(m)}| = 1 - \left[(1 - |\lambda_k|)^{1/k} - \frac{1}{\sqrt{m}}\right]^k + R_k^{(m)}, \ \arg\lambda_k^{(m)} = \arg\lambda_k, \quad k = 1, 2, \ldots \tag{10}$$

It easy to chek that $\lambda_j^{(m)} \neq \lambda_k^{(m)}$ if $j \neq k$ and $\lambda_k^{(m)} \to \lambda_k$, $m \to \infty$, for every k and

$$\sum_{k=1}^{\infty}(1-|\lambda_k^{(m)}|) \to \sum_{k=1}^{n}(1-|\lambda_k|), \quad \text{if } m \to \infty . \tag{11}$$

Let $B_m(z)$, $m \in N$, be the Blaschke product with zeros $(\lambda_k^{(m)})_{k=1}^{\infty}$. It is easy to chek that

$$\lim_{m\to\infty} b_n(\lambda_k^{(m)}) = 0, \quad \lim_{m\to\infty} B_m(\lambda_k) = 0 . \tag{12}$$

Let $r_m(z) = |b_n(z) - B_m(z)|$, $m \in N$. Using (12) it is easy to chek that $r_m(\lambda_k) \to 0$, and $r_m(\lambda_k^{(m)}) \to 0$ if $m \to \infty$, $k = 1, 2, \ldots$ If $z \neq \lambda_k^{(m)}$, and $z \neq \lambda_k^{(m)}$, then also

$$r_m(z) \to 0 \tag{13}$$

uniformly on the compact subsets of the unit disk.

References

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АПРОКСИМАЦИЈА НА КОНЕЧЕН БЛАШКЕОВ ПРОИЗВОД СО БЕСКОНЕЧЕН

Никола Пандески

Р е з и м е

Нека $b_n(z)$ е конечен Блашкеов производ во единичниот диск. Конструеираме низа бесконечни Блахкеови производи која конвергира кон $b_n(z)$ рамномерно на контактни подмножества од единичниот диск.

University "St. Kiril and Metodij"

Institute of Mathematics

P.O. Box 162

1000 Skopje

Republic of Macedonia

e-mail: pandeski@iunona.pmf.ukim.edu.mk