Math. Maced.
Vol. 7 (2009)
13–20

REMARKS ON $\alpha\hat{g}$-HOMEOMORPHISMS

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Abstract. In this paper we introduce new generalizations of homeomorphisms called $\alpha\hat{g}$-homeomorphisms and strongly $\alpha\hat{g}$-homeomorphisms and study some of their properties in topological spaces.

1. Introduction

Njåstad [17] introduced α-open sets. Maki et al.[13] have generalized the concepts of closed sets to α-generalized closed (briefly αg-closed) sets which are strictly weaker than α-closed sets. Veera Kumar [20] defined $\hat{g}$-closed sets. Recently, Abd El Monsef et al.[1] have introduced $\alpha\hat{g}$-closed sets which lie between α-closed sets and αg-closed sets in topological spaces. Maki et al.[14] introduced generalized-homeomorphisms (briefly g-homeomorphisms) and gc-homeomorphisms which are generalizations of homeomorphisms in topological spaces. The purpose of this paper is to introduce $\alpha\hat{g}$-homeomorphisms and strongly $\alpha\hat{g}$-homeomorphisms and investigate some of their properties in topological spaces.

2. Preliminaries

Throughout this paper (X,τ), (Y,σ) and (Z,η) (or simply X, Y and Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A respectively.
We recall the following definitions which are useful in the sequel.

Definition 2.1. *A subset A of a space (X,τ) is called*
(1) *semi-open [10] if $A \subseteq Cl(Int(A))$.*
(2) *α-open [17] if $A \subseteq Int(Cl(Int(A)))$.*

The complement of a semi-open (resp. α-open) set is called semi-closed (resp. α-closed).

The α-closure [17] (resp. semi-closure [5]) of a subset A of X, denoted by $\alpha Cl(A)$ (resp. $sCl(A)$) is defined to be the intersection of all α-closed (resp. semi-closed) sets containing A.
The α-interior [17] of a subset A of X, denoted by $\alpha Int(A)$ is defined to be the

2000 AMS Subject Classification: 54A05, 54D10, 54C08.
Key Words and Phrases: $\alpha\hat{g}$-closed sets, $\alpha\hat{g}$-open sets, $\alpha\hat{g}$-open maps, $\alpha\hat{g}$-continuous maps.

union of all α-open sets contained in A.
For a topological space (X,τ), the subset A of (X,τ) is α-closed (resp. semi-closed) if and only if $\alpha Cl(A) = A$ (resp. $sCl(A) = A$).

Definition 2.2. *A subset A of a space (X,τ) is called*

(1) *generalized closed (briefly g-closed) [11] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).*
(2) *generalized semi-closed (briefly gs-closed) [2] if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).*
(3) *semi-generalized closed (briefly sg-closed) set [4] if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open set in (X,τ).*
(4) *α-generalized closed (briefly αg-closed) [13] if $\alpha Cl(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).*
(5) *$\hat{g}$-closed [20] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X,τ). The complement of $\hat{g}$-closed set is called $\hat{g}$-open.*
(6) *α-$\hat{g}$-closed (briefly $\alpha\hat{g}$-closed) [1] if $\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in (X,τ).*

The complement of a g-closed (resp. gs-closed, sg-closed, αg-closed, $\alpha\hat{g}$-closed) set is called g-open (resp. gs-open, sg-open, αg-open, $\alpha\hat{g}$-open).

Definition 2.3. *A map $f : (X,\tau) \to (Y,\sigma)$ is called*

(1) *α-continuous [16] if $f^{-1}(V)$ is α-closed in (X,τ) for every closed set V in (Y,σ).*
(2) *generalized continuous (briefly g-continuous) [3] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V in (Y,σ).*
(3) *semi-generalized continuous (briefly sg-continuous) [19] if $f^{-1}(V)$ is sg-closed in (X,τ) for every closed set V in (Y,σ).*
(4) *generalized semi-continuous (briefly gs-continuous) [8] if $f^{-1}(V)$ is gs-closed in (X,τ) for every closed set V in (Y,σ).*
(5) *$\alpha\hat{g}$-continuous [9] if $f^{-1}(V)$ is $\alpha\hat{g}$-closed in (X,τ) for every closed set V in (Y,σ).*
(6) *α-closed (resp. α-open) [16] if the image of every closed (resp. open) set in (X,τ) is α-closed (resp. α-open) set in (Y,σ).*
(7) *pre-α-closed (resp. pre-α-open) [6] if $f(F)$ is α-closed (resp. α-open) in (Y,σ) for every α-closed (resp. α-open) set F in (X,τ).*
(8) *generalized closed (briefly g-closed) (resp. g-open) [15] if the image of every closed (resp. open) set in (X,τ) is g-closed (resp. g-open) in (Y,σ).*
(9) *generalized semi-closed (briefly gs-closed) (resp. gs-open) [7] if the image of every closed (resp. open) set in (X,τ) is gs-closed (resp. gs-open) in (Y,σ).*
(10) *semi-generalized closed (briefly sg-closed) (resp. sg-open) [7] if the image of every closed (resp. open) set in (X,τ) is sg-closed (resp. sg-open) in (Y,σ).*

(11) *$\alpha\hat{g}$-closed (resp. $\alpha\hat{g}$-open) [9] if the image of every closed (resp. open) set in (X, τ) is $\alpha\hat{g}$-closed (resp. $\alpha\hat{g}$-open) in (Y, σ).*
(12) *strongly $\alpha\hat{g}$-closed (resp. strongly $\alpha\hat{g}$-open) [18] if the image of every $\alpha\hat{g}$-closed (resp. $\alpha\hat{g}$-open) set in (X, τ) is $\alpha\hat{g}$-closed (resp. $\alpha\hat{g}$-open) in (Y, σ).*
(13) *α-irresolute [12] if $f^{-1}(V)$ is α-open in (X, τ) for every α-open set V in (Y, σ).*
(14) *gc-irresolute [3] if $f^{-1}(V)$ is g-closed in (X, τ) for every g-closed set V in (Y, σ).*
(15) *$\alpha\hat{g}$-irresolute [9] if $f^{-1}(V)$ is an $\alpha\hat{g}$-closed in (X, τ) for every $\alpha\hat{g}$-closed set V in (Y, σ).*
(16) *α-homeomorphism [6] if f is bijective, α-irresolute and pre-α-closed.*
(17) *generalized-homeomorphism (briefly g-homeomorphism) [14] if f is bijective, g-open and g-continuous.*
(18) *semi-generalized homeomorphism (briefly sg-homeomorphism) [8] if f is bijective, sg-open and sg-continuous.*
(19) *generalized semi-homeomorphism (briefly gs-homeomorphism) [8] if f is bijective, gs-open and gs-continuous.*
(20) *gc-homeomorphism [14] if f is bijective, gc-irresolute and it's inverse is gc-irresolute.*

Remark 2.4. *It is easily shown that $f : (X, \tau) \to (Y, \sigma)$ is*
(i). α-irresolute if and only if $f^{-1}(V)$ is α-closed in (X, τ) for every α-closed set V in (Y, σ).
(ii). $\alpha\hat{g}$-irresolute if and only if $f^{-1}(V)$ is an $\alpha\hat{g}$-open in (X, τ) for every $\alpha\hat{g}$-open set V in (Y, σ).

3. $\alpha\hat{g}$-HOMEOMORPHISMS

We introduce the following definition.

Definition 3.1. *A bijective map $f : (X, \tau) \to (Y, \sigma)$ is called an $\alpha\hat{g}$-homeomorphism if f is both $\alpha\hat{g}$-open and $\alpha\hat{g}$-continuous.*

Theorem 3.2. *Every homeomorphism is $\alpha\hat{g}$-homeomorphism.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. Then f is bijective, open and continuous map. Let U be an open set in (X, τ). Since f is an open map, $f(U)$ is an open set in (Y, σ). Every open set is $\alpha\hat{g}$-open and hence $f(U)$ is $\alpha\hat{g}$-open in (Y, σ). This implies f is $\alpha\hat{g}$-open. Let V be a closed set in (Y, σ). Since f is continuous, $f^{-1}(V)$ is closed in (X, τ). Thus $f^{-1}(V)$ is $\alpha\hat{g}$-closed in (X, τ) and therefore, f is $\alpha\hat{g}$-continuous. Hence, f is an $\alpha\hat{g}$-homeomorphism. □

Remark 3.3. *The converse of Theorem 3.2 need not be true as shown in the following example.*

Example 3.4. *Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$, respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is $\alpha\hat{g}$-open and $\alpha\hat{g}$-continuous. Hence f is an $\alpha\hat{g}$-homeomorphism.*

However, $f^{-1}(\{a,c\}) = \{a,c\}$ is not closed in (X,τ) where $\{a,c\}$ is closed in (Y,σ) and hence, f is not continuous. Therefore, f is not a homeomorphism.

Proposition 3.5. *For any bijective map $f : (X,\tau) \to (Y,\sigma)$ the following statements are equivalent.*
(a). $f^{-1} : (Y,\sigma) \to (X,\tau)$ is $\alpha\hat{g}$-continuous map.
(b). f is an $\alpha\hat{g}$-open map.
(c). f is an $\alpha\hat{g}$-closed map.

Proof. (a) $\Rightarrow$ (b). Let U be an open set in (X,τ). Then $X - U$ is closed in (X,τ). Since f^{-1} is $\alpha\hat{g}$-continuous $(f^{-1})^{-1}(X-U)$ is $\alpha\hat{g}$-closed in (Y,σ). That is $f(X-U) = Y - f(U)$ is $\alpha\hat{g}$-closed in (Y,σ). This implies $f(U)$ is $\alpha\hat{g}$-open in (Y,σ). Hence f is $\alpha\hat{g}$-open map.
(b) $\Rightarrow$ (c). Let F be a closed set in (X,τ). Then $X - F$ is open in (X,τ). Since f is $\alpha\hat{g}$-open, $f(X-F)$ is $\alpha\hat{g}$-open in (Y,σ). That is $Y - f(F)$ is $\alpha\hat{g}$-open in (Y,σ). This implies $f(F)$ is $\alpha\hat{g}$-closed in (Y,σ). Hence f is $\alpha\hat{g}$-closed map.
(c) $\Rightarrow$ (a). Let V be closed set in (X,τ). Since f is $\alpha\hat{g}$-closed map, $f(V)$ is $\alpha\hat{g}$-closed in (Y,σ). That is $(f^{-1})^{-1}(V)$ is $\alpha\hat{g}$-closed in (Y,σ). Hence f^{-1} is $\alpha\hat{g}$-continuous. □

Proposition 3.6. *Let $f : (X,\tau) \to (Y,\sigma)$ be a bijective and $\alpha\hat{g}$-continuous map. Then the following statements are equivalent.*
(a). f is an $\alpha\hat{g}$-open map.
(b). f is an $\alpha\hat{g}$-homeomorphism.
(c). f is an $\alpha\hat{g}$-closed map.

Proof. (a) $\Rightarrow$ (b). Let f be an $\alpha\hat{g}$-open map. By hypothesis, f is bijective and $\alpha\hat{g}$-continuous. Thus f is an $\alpha\hat{g}$-homeomorphism.
(b) $\Rightarrow$ (c). Let f be an $\alpha\hat{g}$-homeomorphism. Then f is $\alpha\hat{g}$-open. By the Proposition 3.5, f is $\alpha\hat{g}$-closed.
(c) $\Rightarrow$ (a) is obtained from Proposition 3.5. □

Theorem 3.7. *Every α-homeomorphism is $\alpha\hat{g}$-homeomorphism.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be an α-homeomorphism. Then f is bijective, α-irresolute and pre-α-closed. Let F be closed in (X,τ). Then F is α-closed in (X,τ). Since f is pre-α-closed, $f(F)$ is α-closed in (Y,σ). Every α-closed set is $\alpha\hat{g}$-closed and hence $f(F)$ is $\alpha\hat{g}$-closed in (Y,σ). This implies f is $\alpha\hat{g}$-closed map. Let V be a closed set of (Y,σ). Then V is α-closed in (Y,σ). Since f is α-irresolute $f^{-1}(V)$ is α-closed in (X,τ). Thus $f^{-1}(V)$ is $\alpha\hat{g}$-closed in (X,τ). Therefore f is $\alpha\hat{g}$-continuous. By Proposition 3.5, f is an $\alpha\hat{g}$-homeomorphism. □

Remark 3.8. *The following example shows that the converse of Theorem 3.7 need not be true.*

Example 3.9. *Let $X = \{a,b,c\} = Y$ with topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a,b\}, Y\}$, respectively. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then f is not an α-homeomorphism, because $f(\{b\}) = \{b\}$ is not α-closed in (Y,σ) where $\{b\}$ is α-closed in (X,τ). However, f is an $\alpha\hat{g}$-homeomorphism.*

Remark 3.10. *Next example shows that the composition of two $\alpha\hat{g}$-homeomorphisms is not always an $\alpha\hat{g}$-homeomorphism.*

Example 3.11. *Let $X = Y = Z = \{a,b,c\}$ with topologies $\tau = \{\phi, \{a\}, \{a,c\}, X\}, \sigma = \{\phi, \{a\}, Y\}$ and $\eta = \{\phi, \{a,b\}, Z\}$, respectively. Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be two identity maps. Then both f and g are $\alpha\hat{g}$-homeomorphisms. The set $\{a,c\}$ is open in (X,τ) but $(g \circ f)(\{a,c\}) = \{a,c\}$ is not $\alpha\hat{g}$-open in (Z,η). This implies that $g \circ f$ is not $\alpha\hat{g}$-open and hence $g \circ f$ is not $\alpha\hat{g}$-homeomorphism.*

Theorem 3.12. *Every $\alpha\hat{g}$-homeomorphism is gs-homeomorphism.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be an $\alpha\hat{g}$-homeomorphism. Then f is bijective, $\alpha\hat{g}$-open and $\alpha\hat{g}$-continuous map. Let U be open in (X,τ). Then $f(U)$is $\alpha\hat{g}$-open in (Y,σ). Every $\alpha\hat{g}$-open set is gs-open and hence, $f(U)$is gs-open. This implies f is gs-open map. Let V be closed set in (Y,σ). Then $f^{-1}(V)$ is $\alpha\hat{g}$-closed in (Y,σ). Hence $f^{-1}(V)$ is gs-closed in (Y,σ). This implies f is gs-continuous. Hence f is gs-homeomorphism. □

Remark 3.13. *The following example shows that the converse of Theorem 3.12 need not be true.*

Example 3.14. *Let $X = \{a,b,c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b,c\}, Y\}$, respectively. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then f is gs-homeomorphism, but f is not $\alpha\hat{g}$-homeomorphism because $f(\{b\}) = \{b\}$ is not $\alpha\hat{g}$-open in (Y,σ), where $\{b\}$ is open in (X,τ).*

Remark 3.15. *The following examples show that the concepts of $\alpha\hat{g}$-homeomorphisms and g-homeomorphisms are independent of each other.*

Example 3.16. *Let $X = \{a,b,c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$ and $\sigma = \{\phi, \{b\}, \{a,b\}, Y\}$, respectively. Define a map $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = a$ and $f(c) = c$. Clearly, f is an $\alpha\hat{g}$-homeomorphism. The set $\{a,c\}$ is open in (X,τ) but $f(\{a,c\}) = \{b,c\}$ is not g-open in (Y,σ). This implies f is not g-open map. Therefore f is not g-homeomorphism.*

Example 3.17. *Let $X = \{a,b,c\} = Y$ with topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b,c\}, Y\}$. Define a map $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. The set $\{a\}$ is open in (X,τ) but $f(\{a\}) = \{b\}$ is not $\alpha\hat{g}$-open in (Y,σ). This shows that f is not $\alpha\hat{g}$-open map. Hence f is not $\alpha\hat{g}$-homeomorphism. However, f is g-homeomorphism.*

Remark 3.18. *$\alpha\hat{g}$-homeomorphisms and sg-homeomorphisms are independent of each other as shown below. The function f defined in Example 3.16 is $\alpha\hat{g}$-homeomorphism. $f^{-1}(\{a,c\}) = \{b,c\}$ is not sg-closed in (X,τ) where $\{a,c\}$ is closed in (Y,σ) implies f is not sg-continuous and hence, f is not a sg-homeomorphism.*

Example 3.19. *Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$, respectively. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b, f(b) = a$ and $f(c) = c$. Clearly, f is a sg-homeomorphism. The set $\{a\}$ is open in (X, τ) but $f(\{a\}) = \{b\}$ is not $\alpha\hat{g}$-open in (Y, σ). This implies f is not $\alpha\hat{g}$-open map and hence f is not $\alpha\hat{g}$-homeomorphism.*

4. Strongly $\alpha\hat{g}$-homeomorphisms

Definition 4.1. *A bijection $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly $\alpha\hat{g}$-homeomorphism if f is $\alpha\hat{g}$-irresolute and its inverse f^{-1} is also $\alpha\hat{g}$-irresolute.*

Theorem 4.2. *Every strongly $\alpha\hat{g}$-homeomorphism is $\alpha\hat{g}$-homeomorphism.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be strongly $\alpha\hat{g}$-homeomorphism. Let U be open in (X, τ). Then U is $\alpha\hat{g}$-open in (X, τ). Since f^{-1} is $\alpha\hat{g}$-irresolute, $(f^{-1})^{-1}(U)$ is $\alpha\hat{g}$-open in (Y, σ). That is $f(U)$ is $\alpha\hat{g}$-open in (Y, σ). This implies f is $\alpha\hat{g}$-open map. Let F be closed in (Y, σ). Then F is $\alpha\hat{g}$-closed in (Y, σ). Since f is $\alpha\hat{g}$-irresolute, $f^{-1}(F)$ is $\alpha\hat{g}$-closed in (X, τ). This implies f is $\alpha\hat{g}$-continuous map. Hence f is $\alpha\hat{g}$-homeomorphism. □

Remark 4.3. *The following example shows that the converse of Theorem 4.2 need not be true.*

Example 4.4. *Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Clearly, f is an $\alpha\hat{g}$-homeomorphism. The set $\{a, c\}$ is $\alpha\hat{g}$-closed in (Y, σ), but $f^{-1}(\{a, c\}) = \{a, c\}$ is not $\alpha\hat{g}$-closed in (X, τ). So, f is not $\alpha\hat{g}$-irresolute and hence f is not strongly $\alpha\hat{g}$-homeomorphism.*

Theorem 4.5. *The composition of two strongly $\alpha\hat{g}$-homeomorphism is strongly $\alpha\hat{g}$-homeomorphism.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two strongly $\alpha\hat{g}$-homeomorphisms. Let F be an $\alpha\hat{g}$-closed set in (Z, η). Since g is $\alpha\hat{g}$-irresolute map, $g^{-1}(F)$ is $\alpha\hat{g}$-closed in (Y, σ). Since f is $\alpha\hat{g}$-irresolute, $f^{-1}(g^{-1}(F))$ is $\alpha\hat{g}$-closed in (X, τ). That is $(g \circ f)^{-1}(F)$ is $\alpha\hat{g}$-closed in (X, τ). This implies that $g \circ f : (X, \tau) \to (Z, \eta)$ is $\alpha\hat{g}$-irresolute . Let V be an $\alpha\hat{g}$-closed in (X, τ). Since f^{-1} is $\alpha\hat{g}$-irresolute $(f^{-1})^{-1}(V)$ is $\alpha\hat{g}$-closed in (Y, σ). That is $f(V)$ is $\alpha\hat{g}$-closed in (Y, σ). Since g^{-1} is $\alpha\hat{g}$-irresolute, $(g^{-1})^{-1}(f(V))$ is $\alpha\hat{g}$-closed in (Z, η). That is $g(f(V))$ is $\alpha\hat{g}$-closed in (Z, η). So, $(g \circ f)(V)$ is $\alpha\hat{g}$-closed in (Z, η). This implies $((g \circ f)^{-1})^{-1}(V)$ is $\alpha\hat{g}$-closed in (Z, η). This shows that $(g \circ f)^{-1} : (Z, \eta) \to (X, \tau)$ is $\alpha\hat{g}$-irresolute. Hence $g \circ f$ is strongly $\alpha\hat{g}$-homeomorphism. □

We denote the family of all strongly $\alpha\hat{g}$-homeomorphisms of a topological space (X, τ) onto itself by $s\alpha\hat{g}$-$h(X, \tau)$.

Theorem 4.6. *The set $s\alpha\hat{g}$-$h(X, \tau)$ is a group under the composition of mappings.*

Proof. By Theorem 4.5, $g \circ f \in s\alpha\hat{g}\text{-}h(X,\tau)$ for all $f, g \in s\alpha\hat{g}\text{-}h(X,\tau)$. It is known that the composition of mappings is associative. The identity map $I : (X,\tau) \to (X,\tau)$ belonging to $s\alpha\hat{g}\text{-}h(X,\tau)$ serves as the identity element. If $f \in s\alpha\hat{g}\text{-}h(X,\tau)$, then $f^{-1} \in s\alpha\hat{g}\text{-}h(X,\tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $s\alpha\hat{g}\text{-}h(X,\tau)$. Hence, $s\alpha\hat{g}\text{-}h(X,\tau)$ is a group under the composition of mappings. □

Theorem 4.7. *Let $f : (X,\tau) \to (Y,\sigma)$ be a strongly $\alpha\hat{g}$-homeomorphism. Then f induces an isomorphism from the group $s\alpha\hat{g}\text{-}h(X,\tau)$ onto the group $s\alpha\hat{g}\text{-}h(Y,\sigma)$.*

Proof. Using the map f, we define a map $\theta_f : s\alpha\hat{g}\text{-}h(X,\tau) \to s\alpha\hat{g}\text{-}h(Y,\sigma)$ by $\theta_f(k) = f \circ k \circ f^{-1}$ for every $k \in s\alpha\hat{g}\text{-}h(X,\tau)$. Then θ_f is a bijection. Further, for all $k_1, k_2 \in s\alpha\hat{g}\text{-}h(X,\tau)$, $\theta_f(k_1 \circ k_2) = f \circ (k_1 \circ k_2) \circ f^{-1} = (f \circ k_1 \circ f^{-1}) \circ (f \circ k_2 \circ f^{-1}) = \theta_f(k_1) \circ \theta_f(k_2)$. Therefore θ_f is a homeomorphism and so it is an isomorphism induced by f. □

Remark 4.8. *Strongly $\alpha\hat{g}$-homeomorphisms and α-homeomorphisms are independent notions as shown in the following examples.*

Example 4.9. *Let $X = \{a, b, c\} = Y$, with topologies $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$, respectively. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Clearly f is strongly $\alpha\hat{g}$-homeomorphism. $f^{-1}(\{a, c\}) = \{a, c\}$ is not α-closed in (X,τ) where $\{a, c\}$ is α-closed in (Y,σ). This shows that f is not α-irresolute and hence f is not α-homeomorphism.*

Example 4.10. *Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity map.*
$f^{-1}(\{a, b\}) = \{a, b\}$ is not $\alpha\hat{g}$-closed in (X,τ) where $\{a, b\}$ is $\alpha\hat{g}$-closed in (Y,σ). This implies f is not $\alpha\hat{g}$-irresolute and hence f is not strongly $\alpha\hat{g}$-homeomorphism. However, f is an α-homeomorphism.

Remark 4.11. *The concepts of strongly $\alpha\hat{g}$-homeomorphisms and gc-homeomorphisms are independent of each other as the following examples show.*

Example 4.12. *Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then f is strongly $\alpha\hat{g}$-homeomorphism, but it is not gc-homeomorphism because $f^{-1}(\{b\}) = \{b\}$ is not g-closed in (X,τ), where $\{b\}$ is g-closed in (Y,σ).*

Example 4.13. *Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity map. The set $\{b\}$ is $\alpha\hat{g}$-closed in (Y,σ), but $f^{-1}(\{b\}) = \{b\}$ is not $\alpha\hat{g}$-closed in (X,τ). This implies f is not $\alpha\hat{g}$-irresolute and hence f is not strongly $\alpha\hat{g}$-homeomorphism. However, f is a gc-homeomorphism.*

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