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ON SOLVING A NONLINEAR ORDINARY DIFFERENTIAL EQUATION OF THE THIRD ORDER

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Abstract

In this note a class of integrable nonlinear ordinary differential equation of the third order is given.

The object of this note is investigation of integrability of the differential equation of the form

$$y''' + a(x)y'' + b(x)y' + A(y)y'^3 + a(x)$$

$$B(y)y'^2 + C(y)y'y'' + c(x)D(y) = 0\,. \qquad (1)$$

The same equation is treated in paper [1]. Namely, J. D. Kečkić proved the following result:

If $u = u(x)$ is a solution of the equation

$$u''' + a(x)u'' + b(x)u' + c(x)u = 0\,. \qquad (2)$$

the equation (1), where

$$A(y) = B(y)^2 + B'(y), \quad C(y) = 3B(y), \quad D'(y) + D(y)B(y) = 1\,, \qquad (3)$$

has a solution $y = G(u(x))$. The function G is any solution of the equation $G' - w(G) = 0$ where

$$B(y) = -\frac{w'(y)}{w(y)}\,.$$

We will use procedure developed by He Ruhong and Fan Xing [3] to study a class of the equation (1). Notice that the nonlinear ordinary differential equations of the second order are considered in [3].

We start from the equation (2) where $a(x)$, $b(x)$, $c(x)$ are to be determined later. By the transformation $u = g(y)$ this equation becomes

$$y'''+a(x)y''+b(x)y'+\frac{g'''(y)}{g'(y)}y'^3+a(x)\frac{g''(y)}{g'(y)}y'^2+\frac{3g''(y)}{g'(y)}y'y''+c(x)\frac{g(y)}{g'(y)}=0. \quad (4)$$

According to V. Lj. Kocić [2] the general solution of the equation (4) is given by $y = g^{-1}(C_1u_1 + C_2u_2 + C_3u_3)$ where u_1, u_2, u_3 are linearly independent solutions of the equation (2).

If we introduce the substitution $t = f(x)$, $u(x) = z(t)$, where z and t are new unknown function and new variable, respectively, the equation (2) becomes

$$f'(x)^3z'''(t) + \big(3f'(x)f''(x) + a(x)f'(x)^2\big)z''(t)+$$
$$+\big(f'''(x) + a(x)f''(x) + b(x)f'(x)\big)z'(t) + c(x)z(t) = 0\,.$$

From

$$3f'(x)f''(x) + a(x)f'(x)^2 = 0\,,$$
$$f'''(x) + a(x)f''(x) + b(x)f'(x) = 0$$

we have

$$a(x) = -\frac{3f''(x)}{f'(x)}, \qquad b(x) = 3\left(\frac{f''(x)}{f'(x)}\right)^2 - \frac{f'''(x)}{f'(x)}\,. \quad (5)$$

Furthermore, from

$$f'(x)^3z'''(t) + c(x)z(t) = 0\,, \qquad F(t)z'''(t) - F'''(t)z(t) = 0 \quad (6)$$

we have

$$c(x) = -\frac{F'''(f(x))f'(x)^3}{F(f(x))}\,. \quad (7)$$

It is known that the general solution of the equation (6) is given by

$$z(t) = C_1F(t) + C_2\int h_1(t)dt + C_3\int h_2(t)dt$$

where $h_1(t)$, $h_2(t)$ are linearly independent solutions of the equation

$$F(t)h''(t) + 3F'(t)h'(t) + 3F''(t)h(t) = 0\,. \quad (8)$$

Substituting (5) and (7) into (4) we obtain the equation

$$y''' - \frac{3f''(x)}{f'(x)}y'' + 3\left[\left(\frac{f''(x)}{f'(x)}\right)^2 - \frac{f'''(x)}{f'(x)}\right]y' + \frac{g'''(y)}{g'(y)}y'^3 -$$
$$-\frac{3f''(x)}{f'(x)}\frac{g''(y)}{g'(y)}y'^2 + \frac{3g''(y)}{g'(y)}y'y'' - \frac{F'''\big(f(x)\big)f'(x)^3}{F\big(f(x)\big)}\frac{g(y)}{g'(y)} = 0 \quad (9)$$

or the equation (1) where

$$b(x) = \frac{2}{9}a(x)^2 + \frac{1}{3}a'(x)\,,$$

$$c(x) = -\frac{F'''(\int e^{-\frac{1}{3}\int a(x)\,dx}\,dx)e^{-\int a(x)\,dx}}{F(\int e^{-\frac{1}{3}\int a(x)\,dx}\,dx)}$$

(F is arbitrary function) and (3).

In accordance with the above and [2] we can formulate the following result.

Theorem. Suppose that $f, g, F \in C^3(I)$, $f, g \neq$ const., $F \neq 0$ and the function g has the inverse function g^{-1}. Then the general solution of the equation (9) is given by

$$y = g^{-1}\big(C_1 F(f(x)) + C_2 \int h_1(f(x))\, df(x) + C_3 \int h_2(f(x))\, df(x)\big) \quad (10)$$

where $h_1(t)$, $h_2(t)$ are linearly independent solutions of the equation (8).

Example. The equation

$$y''' + \frac{6x}{1+x^2}\, y'' + \frac{3(x^2-1)}{(1+x^2)^2}\, y' + 2y^{-2}y'^3 - \frac{12x}{1+x^2}\, y^{-2}y'^2 - 3y^{-1}y'y'' -$$

$$- \frac{6}{(1+x^2)^2 \operatorname{arctg} x}\, y \ln y = 0 \quad (y > 0)$$

has the general solution

$$y = \exp\Big(C_1(\operatorname{arctg} x)^3 + C_2 \int \frac{\cos(\sqrt{2}\ln \operatorname{arctg} x)}{(1+x^2)(\operatorname{arctg} x)^4}\, dx +$$

$$+ C_3 \int \frac{\sin(\sqrt{2}\ln \operatorname{arctg} x)}{(1+x^2)(\operatorname{arctg} x)^4}\, dx\Big) \quad (x > 0).$$

Here $g(y) = \ln y$, $F(t) = t^3$ and $f(x) = \operatorname{arctg} x$. To the equation (8) coresponds the equation

$$t^2 h''(t) + 9t\, h'(t) + 18\, h(t) = 0$$

and its the general solution is given by

$$h(t) = C_1 t^{-4} \cos(\sqrt{2}\ln t) + C_2 t^{-4} \sin(\sqrt{2}\ln t) \quad (t > 0).$$

References

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ЗА РЕШАВАЊЕ НА ЕДНА ОБИЧНА ДИФЕРЕНЦИЈАЛНА РАВЕНКА ОД ТРЕТИ РЕД

Здравко Ф. Старц

Р е з и м е

Во овој труд е докажан следниот резултат:

Теорема. Нека е f, g, $F \in C^3(I)$, f, $g \neq$ const., $F \neq 0$ и нека функцијата g има инверзна функција g^{-1}. Тогаш општото решение на равенката (9) е дадено со (10) каде што $h_1(t)$, $h_2(t)$ се линеарно независни решенија на равенката (8).

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