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AN IMPROVEMENT OF JENSEN'S INEQUALITY

Sever S. Dragomir

Abstract. A refinement of the well-known inequality due to Jensen and some natural applications are given.

The main aim of this note is to point out a refinement of the well-known discrete inequality due to Jensen:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

where $f: C \subseteq X \to \mathbb{R}$ is a convex mapping on convex set C of real linear space X, x_i are in C $(i=1,\dots,n)$ and $p_i \ge 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$.

The following theorem holds.

Theorem. Let f, C, x_i, p_i be as above. Then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(tx_i+(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \le$$

$$\le \frac{1}{P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j f(tx_i+(1-t)x_j) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{2}$$

for all t in $[0,1]$.

Proof. Firstly, we observe that

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(tx_i+(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) =$$

$$= \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\frac{1}{P_n}\sum_{j=1}^{n} (tx_i+(1-t)x_j)p_j\right).$$

Applying Jensen's inequality, we derive that

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\frac{1}{P_n}\sum_{j=1}^{n} (tx_i+(1-t)x_j)p_j\right) \ge$$

$$\ge f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\left[\frac{1}{P_n}\sum_{j=1}^{n} (tx_i+(1-t)x_j)p_j\right]\right).$$

Since simple computation shows that

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i\left[\frac{1}{P_n}\sum_{j=1}^{n}(tx_i+(1-t)x_j)p_j\right] = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$$

the first inequality in (2) is obtained.

To prove the second inequality, we also use Jensen's result. Namely, we have:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f\Big(\frac{1}{P_n}\sum_{j=1}^{n} p_j(tx_i+(1-t)x_j)\Big) \le$$

$$\le \frac{1}{P_n}\sum_{i=1}^{n} p_i\left[\frac{1}{P_n}\sum_{j=1}^{n} f(tx_i+(1-t)x_j)p_j\right] =$$

$$= \frac{1}{P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(tx_i+(1-t)x_j)p_i p_j .$$

Finally, for all $i,j\in\{1,\dots,n\}$, we have:

$$f(tx_i+(1-t)x_j) \le tf(x_i) + (1-t)f(x_j).$$

Multiplying this inequality with $p_i p_j \ge 0$ and summing after i and j to 1 at n, we derive:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(tx_i+(1-t)x_j)p_i p_j \le \sum_{i=1}^{n}\sum_{j=1}^{n}\big[tf(x_i) +$$

$$+ (1-t)f(x_j)\big]p_i p_j = P_n\sum_{i=1}^{n} p_i f(x_i),$$

and the proof is finished.

<u>Corollary</u>. In the above assumptions we have:

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f\Big(\frac{1}{2}x_i + \frac{1}{2}\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\Big) \le$$

$$\le \frac{1}{P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j\, f\Big(\frac{x_i+x_j}{2}\Big) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

<u>Applications</u>. <u>1</u>. Let $(X, \|\cdot\|)$ be a normed linear space, $p \ge 1$, $x_i\in X$, $p_i \ge 0$ $(i=1,\dots,n)$ with $P_n > 0$. Then

$$\Big\|\sum_{i=1}^{n} p_i x_i\Big\|^p \le P_n^{p-1}\sum_{i=1}^{n} p_i \Big\|tx_i+(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\Big\|^p \le$$

$$\le P_n^{p-2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \|tx_i+(1-t)x_j\|^p \le P_n^{p-1}\sum_{i=1}^{n} p_i \|x_i\|^p.$$

2. Let $x_i, p_i \geq 0$ $(i=1,\ldots,n)$ with $P_n > 0$. Then the following refinement of arithmetic-geometric means' inequality holds

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \geq \left[\prod_{i=1}^{n}\left(tx_i+(1-t)\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)^{p_i}\right]^{1/P_n} \geq$$

$$\geq \left[\prod_{i=1}^{n}\prod_{j=1}^{n}(tx_i+(1-t)x_j)^{p_i p_j}\right]^{1/P_n^2} \geq \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}.$$

The proofs of these inequalities follow from the above theorem for the convex mappings: $f_1: X \to R$, $f_1(x) = \|x\|^p$ and $f_2: (0,\infty) \to R$, $f_2(x) = -\ln x$. We omit the details.

For other recent refinements of Jensen's inequality see the papers [1] and [2] where further applications are given.

R E F E R E N C E S

[1] Dragomir S.S.: A refinement of Jensen inequality, G.M.Metod., 10 (1989), 190-191

[2] Pečarić J.E., and Dragomir S.S.: A refinement of Jensen inequality and applications, Studia Univ. Babes-Bolyai, Mathematica, 34 (1)(1989), 15-19

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Str. Bega Nr. 1

1700 Reşita

Jud. Caraş-Severin

România