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DIFFERENTIATION OF FUNCTIONS ON VILENKIN GROUPS*

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Abstract

In this paper we will consider a concept of differentiation of complex functions defined on Vilenkin groups, looking from the aspect of summability of Vilenkin series. In that sense we give matrix interpretation of C. N. Onneweer's concept of differentiation [5, Definition 3], and then our generalization of that concept. We also give main properties of this generalized derivative. Moreover, we show that every method of summation of Vilenkin series, in a certain way, defines some concept of differentiation of integrable functions on Vilenkin groups.

1. Introduction and preliminaries

Vilenkin group G is an infinite compact totally disconnected Abelian group whose topology satisfies the second axiom of countability. Vilenkin [8] has proved that topology in G can be given by a basic chain of neighborhoods of zero

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots \supset \{0_G\}, \quad \bigcap_{n \in N_0} G_n = \{0_G\}, \quad (1.1)$$

consisting of open subgroups of a group G, such that the factor group G_n/G_{n+1} is a cyclic group of a prime order p_{n+1}, for every $n \in \boldsymbol{N}_0$. We shall call a group G **bounded** iff a sequence (p_n) is bounded. There exists the normalized Haar measure μ on G, such that

$$\mu(G_n) = m_n^{-1} (\forall n \in \boldsymbol{N}_0), \quad \text{where} \quad m_n := p_1 \cdot p_2 \cdot \ldots \cdot p_n \ (m_0 := 1). \ (1.2)$$

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Every non-negative integer n has a unique representation

$$n = \sum_{k=1}^{N} a_k m_k\,, \quad N = N(n)\,, \quad \text{where each } a_i \ (i = 0, 1, 2, \dots, N) \tag{1.3}$$

is an integer that satisfies the condition $0 \le a_i \le p_{i+1} - 1$, $a_N \ne 0$.

For $n \in \boldsymbol{N}_0$, let us choose a $g_n \in G_n \setminus G_{n+1}$. As p_n $(n = 1, 2, \dots)$ are primes, everi $g \in G$ can be represented in a unique way as

$$g = \sum_{n=0}^{\infty} a_n g_n\,, \quad \text{for some integers} \quad a_n \in \{0, 1, 2, \dots, p_{n+1} - 1\}\,. \tag{1.4}$$

Then for every $n \in \boldsymbol{N}_0$ holds

$$G_n = \Big\{ g \in G : g = \sum_{i=0}^{\infty} a_i g_i\,, \quad a_i = 0, \quad 0 \le i \le n - 1 \Big\}\,. \tag{1.5}$$

For every $1 \le p \le \infty$ let $\boldsymbol{L}^p(G)$ denote the $\boldsymbol{L}^p$ space on G with respect to the measure μ. The set of continuous functions defined on G with values in $\boldsymbol{C}$ (the set of complex numbers) will be denoted by $\boldsymbol{C}(\boldsymbol{G})$.

Remark 1.1. If $1 \le p_1 < p_2 < \infty$, then $\boldsymbol{L}^{p_2}(\boldsymbol{G}) \subset \boldsymbol{L}^{p_1}(\boldsymbol{G})$. Let Γ denote the group of characters of the group G, and let $G_n^{\perp}$ be the annihilator of the group G_n in Γ. Vilenkin has proved [8] that there exists Paley-type ordering of elements in Γ:
let us choose a $\chi \in G_{k+1}^{\perp} \setminus G_k^{\perp}$ and denote it by χ_{m_k}; then assign to every n, represented by (1.3), the character χ_n defined by

$$\chi_n = \prod_{k=0}^{N} \chi_{m_k}^{a_k}\,. \tag{1.6}$$

We easily see that

$$G_n^{\perp} = \{\chi_j : 0 \le j < m_n\} \quad (\forall n \in \boldsymbol{N}_0)\,. \tag{1.7}$$

$(\chi_n)_{n \in \boldsymbol{N}_0}$ supplied with the above ordering is called a **Vilenkin system.** This system is **bounded** iff the group G is bounded. Dual group Γ of the group G is a discrete, countable Abelian group with torsion ([4]. (24.15) and (24.26)).

Vilenkin series $\sum_{n=0}^{\infty} c_n \chi_n$ is a Fourier series iff there exists a function $f \in L^1(G)$ such that

$$c_n = \hat{f}(n) := \int_0 f \overline{\chi}_n \quad (\forall n \in \boldsymbol{N}_0)\,, \tag{1.8}$$

where $\overline{z}$ denotes the complex-conjugate of z. In that case, the **n-th partial sum** of the series is given by

$$S_n(f) = \sum_{k=0}^{n-1} \hat{f}(k)\chi_k = f^* D_n \,, \tag{1.9}$$

where $D_n := \sum_{k=0}^{n-1} \chi_k$ is the **Dirihlet kernel** of index n on G, and

$$f * \varphi(x) = \int_G f(x-t)\varphi(t)d\mu(t)$$

is the **convolution of functions** f and φ on G.

Chronologically, firstly J. E. Gibs ([2] and [3]), introduced the **DYADIC DERIVATIVE** "[1]" with the following property:

$$[\omega_k(x)]^{[1]} = k\omega_k(x) \,,$$

where $\omega_k(x)$ is the Walsh (J. L. Walsh) function of index k.

This derivative was furtner studied by P. L. Butzer and H. J. Wagner [1], and also F. Schipp [7] who proved that $ka_k \downarrow 0$ yields

$$\left(\sum_{k=0}^{\infty} a_k \omega_k(x)\right)^{[1]} = \sum_{k=0}^{\infty} k a_k \omega_k(x) \,.$$

V. A. Skvortsov and W. R. Wade have proved the analogue result for the series over arbitrary system of characters od 0-dimensional groups under more general assumptions and have simplified the proof.

J. Pal and P. Simon [6] have defined a derivative of a function defined on an arbitrary 0-dimensional compact commutative group. C. W. Onnewer [5] has studied differentiation of functions (with complex values) defined on a dyadic group D. In [5] he has given three definitions of dyadic differentiation where Leibniz differentiation formula does not hold. His main idea was that the derivative on a dyadic group should be defined in such a way that relations between a function defined on D (mainly relations between characters on D) and its derivative be as simple and natural as possible.

For example, the natural relation between the character

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

on the torus group $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ and its derivative $(e^{ikx})' = ike^{ikx}$ should be in some way preserved for a dyadic derivative of a character on D,

In our paper we will start with the following facts from [5]:

Defonition 1.A. ([5, Definition 3], applied to Vilenkin groups).
For a Vilenkin group G, function $f \in L^1(G)$ and $n \in \boldsymbol{N}$ let us define

$$E_n f(x) := \sum_{i=-1}^{n-1} (m_{i+1} - m_i)\big[f(x) - S_{m_i} f(x)\big] \,, \tag{1.10}$$

where $m_{-1} := 0$ and

$$S_{m_i} f(x) = D_{m_i} * f(x) = \sum_{s=0}^{m_i - 1} \hat{f}(s)\chi_s(x)$$

(the partial sum of an index m_i of the Fourier series $\sum_{k=0}^{\infty} \hat{f}(k)\chi_k(x)$ of a function f).

If $\lim_{n\to\infty} E_n f(x)$ exists, than its value is caled **derivation of a function f at $x \in G$**, and denoted by $f^{[1]}(x)$.

Theorem 1.A. [5, *Theorem* 3].

a) *If $\chi \in \Gamma$ and $x \in G$, then $\chi^{[1]}(x)$ exists, and $\chi^{[1]}(x) = \|\chi\| \cdot \chi(x)$, where*

$$\|\chi\| := \begin{cases} m_k, & \text{if } \chi \in G_k^{\perp} \setminus G_{k-1}^{\perp}, \quad \text{for some } k \in \boldsymbol{N} \\ 0, & \text{if } \chi = \chi_0 \end{cases} \tag{1.11}$$

b) *if $f \in L^1(G)$ and $n \in \boldsymbol{N}$, then*

$$(E_n f)^{\wedge}(\chi) = \begin{cases} \|\chi\| \hat{f}(\chi), & \text{if } \chi \in G_n^{\perp} \\ m_n \cdot \hat{f}(\chi), & \text{if } \chi \notin G_n^{\perp}. \end{cases} \tag{1.12}$$

Theorem 1.B. [5, *Theorem* 4]. *If $f \in W\big(L^1(G), \|\chi\|\big)$ then $f \in \mathcal{D}$, where $\mathcal{D}$ is a domain of the differentiation operator defined in Definition 1.A. and*

$$\begin{aligned} W\big(L^1(G), \|\chi\|\big) := \{f \in L^1(G) : \big(\exists g \in L^1(G)\big) \text{ such that} \\ \hat{g}(\chi) = \|\chi\| \hat{f}(\chi), \ \chi \in \Gamma\}. \end{aligned} \tag{1.13}$$

If we define

$$L_k f(x) \equiv L_k(f, x) := S_{m_n} f(x) = \sum_{s=0}^{m_n - 1} \hat{f}(s)\chi_s(x) \tag{1.14}$$

for every $k \in h_n := \{m_n, m_n + 1, \ldots, m_{n+1} - 1\}$ $(n \in \boldsymbol{N}_0)$, then we can write

$$\begin{aligned} E_n f(x) \equiv E_n(f, x) &= \sum_{i=-1}^{n-1} (m_{i+1} - m_i)\big[f(x) - L_{m_i}(f, x)\big] = \\ &= \sum_{i=-1}^{n-1} \sum_{k=m_i}^{m_{i+1}-1} \big[f(x) - L_k(f, x)\big] \quad (\forall n \in \boldsymbol{N}), \end{aligned}$$

Relation (1.13) has a following **matrix interpretation:**

$$
\begin{array}{c}
\\ \\ m_n \rightarrow \\ \\ \\ \\ m_{n+1} \rightarrow \\ \\ \\
\end{array}
\overset{\qquad m_n-1 \;\; m_{n+1}-1 \atop \qquad\quad \downarrow \qquad\quad \downarrow}{
\begin{bmatrix}
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
1\,1\,1 \cdots 1\,0 \cdots 0\,0 \cdots \\
1\,1\,1 \cdots 1\,0 \cdots 0\,0 \cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
1\,1\,1 \cdots 1\,0 \cdots 0\,0 \cdots \\
1\,1\,1 \cdots 1\,1 \cdots 1\,0 \cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{bmatrix}}
\begin{bmatrix}
\cdot \\ \cdot \\ \hat{f}(m_n)\chi_{m_n} \\ \cdot \\ \cdot \\ \cdot \\ \hat{f}(m_{n+1})\chi_{m_{n+1}} \\ \cdot \\ \cdot
\end{bmatrix}
=
\begin{bmatrix}
\vdots \\ \sum_{s=0}^{m_n-1} 1\cdot\hat{f}(s)\chi_s \\ \cdot \\ \sum_{s=0}^{m_n-1} 1\cdot\hat{f}(s)\chi_s \\ \sum_{s=0}^{m_{n+1}-1} 1\cdot\hat{f}(s)\chi_s \\ \vdots
\end{bmatrix}
$$

In that way we have obtained matrix

$$
\Lambda = [\lambda_{ij}](i,j \in \boldsymbol{N}_0), \quad \text{where}
$$

$$
\lambda_{ij} := \begin{cases} 1, & \text{for} \quad m_n \le i < m_{n+1} \wedge 0 \le j < m_n, \quad n \in \boldsymbol{N}_0 \\ 0, & \text{otherwise.} \end{cases} \tag{1.16}
$$

So, we can write

$$
L_n f(x) \equiv L_n(f,x) = \sum_{k=0}^{n-1} \lambda_{nk}\hat{f}(k)\chi_k(x), \forall \in \boldsymbol{N}, \forall x \in G, \big(L_0(f,x) := 0, \forall x \in G\big). \tag{1.17}
$$

This matrix motivates us to introduce the following definition.

Definition 1.1. Let G be a given Vilenkin group, and let $\Lambda = [\lambda_{nk}]$ $(n,k \in \boldsymbol{N}_0)$ be a scalar matrix. For $f \in L^1(G)$ let

$$
L_k(f,\Lambda,x) := \sum_{s=0}^{\infty} \lambda_{ks}\hat{f}(s)\chi_s(x), \quad (k \in \boldsymbol{N}_0) \quad \text{and}
$$

$$
E_N(f,\Lambda,x) := \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} \big[\sigma(f,\Lambda,x) - L_i(f,\Lambda,x)\big] \; (\forall N \in \boldsymbol{N})
$$

(with the condition that $L_k(f,\Lambda,x) \rightarrow \sigma(f,\Lambda,x)$, $k \rightarrow \infty$). Then the limit $\lim\limits_{N\rightarrow\infty} E_N(f,\Lambda,x)$ (if it exists) is called the Λ-derivative of the function f at $x \in G$ and denoted by $f^{\Lambda}(x)$.

If $\lim_{N\to\infty} E_N(f,\Lambda,x) = f^\Lambda(x)$ uniformly on $A \subseteq G$, then the function f^Λ is called **uniform Λ-derivative** of f on A. We will use the following notation:

$$\mathcal{D}(\Lambda,x) = \{f \in L^1(G): \text{ there exists } f^\Lambda(x)\},$$
$$\mathcal{D}(\Lambda,A) = \{f \in L^1(G): \text{ there exists } f^\Lambda(x), \text{ for a.e. } x \in A \subseteq G\},$$

and

$$\mathcal{UD}(\Lambda,A) = \{f \in L^1(G): f \text{ is uniformly } \Lambda\text{-differentiable on } A \subseteq G\}.$$

Remark 1.2.

a) If we take the matrix (1.16) in Definition 1.1, then for every $i \in h_k$ and every $k \in N_0$ we have

$$L_i(f,\Lambda,x) = L_{m_k}(f,\Lambda,x) = \sum_{s=0}^{m_k-1} 1\cdot \hat{f}(s)\chi_s(x) = S_{m_k}f(x).$$

This implies that $L_i(f,\Lambda,x) \to f(x) = \sigma(f,\Lambda,x)$ a.e. on G.

In this case $E_N(f,\Lambda,x) = \sum_{k=-1}^{N-1}(m_{k+1}-m_k)[f(x) - S_{m_k}f(x)]$, so we have $f^\Lambda(x) = \lim_{N\to\infty} E_N(f,\Lambda,x) = f^{[1]}(x)$ and this is exactly Onneweer's concept of diferentiation. Let us notice that for every function $f \in L^p(G)$ $(1 \le p \le \infty)$

$$\|S_{m_n} - f\|_p \to 0 \ (n \to \infty).$$

b) If we take a triangular matrix $[\lambda_{nk}]$ $(\lambda_{nk} = 0,\ k > n)$, in Definition 1.1 then

$$L_n(f,\Lambda,x) = \sum_{k=0}^{n} \lambda_{nk}\hat{f}(k)\chi_k(x), \quad (n \in N_0).$$

c) From definition 1.1 we can see that every matrix $\Lambda = [\lambda_{nk}]$ $(n,k \in N_0)$ that sums the series $\sum_{s=0}^{\infty} \hat{f}(s)\chi_s(x)$ towards $\sigma(f,\Lambda,x)$ defines a Λ-derivative of a function $f \in L^1(G)$ at the point x.

2. Results

Main results of this paper are the following statements about main properties of Λ-derivative of integrable functions on a Vilekin group G.

Theorem 2.1. *Let $G, \Lambda, f, L_n(f,\Lambda,x)$ and $E_N(f,\Lambda,x)$ be as in Definition* 1.1. *Then the following statements hold:*

(i) $L_K(\chi_j,\Lambda,x) = \lambda_{kj}\chi_j(x)$ $(\forall k \in \boldsymbol{N}_0,\ \forall j \in \boldsymbol{N}_0,\ \forall_{\chi_j} \in \Gamma,\ \forall x \in G)$.

(ii) $E_N(\chi_j,\Lambda,x) = \chi_j(x)\Lambda_N(j)$ $(\forall n \in \boldsymbol{N},\ \forall j \in \boldsymbol{N}_0,\ \forall x \in G)$, *where*

$$\Lambda_N(j) := \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} (\lambda_{\infty j} - \lambda_{ij}), \quad \lambda_{\infty j} := \lim_{k\to\infty} \lambda_{kj}.$$

(iii) *For arbitrary $j \in \boldsymbol{N}_0$ χ_j is a Λ-differentiable function at every $x \in G$ iff the limit $\lim_{N\to\infty} \Lambda_N(j) := \Lambda_\infty(j)$ exists. In that case, $\chi_j^\Lambda(x) = \chi_j(x)\Lambda_\infty(j)$ holds.*

(iv) $[L_k(f,\Lambda,x)]^\wedge(j) \equiv \hat{L}_k(j) = \lambda_{kj}\hat{f}(j)$ $(\forall j \in \boldsymbol{N}_0)$, *under the condition that the series $L_k(f,\Lambda,x) \in L^1(G)$ is uniformly convergent on G.*

(v) *If $L^1(G) \ni L_k(f,\Lambda,x) \to \sigma(f,\Lambda,x)$ uniformly on G, then for $N \in \boldsymbol{N}$ and every $j \in \boldsymbol{N}_0$ $[E_N(f,\Lambda,x)]^\wedge(j) \equiv \hat{E}_N(j) = \hat{f}(j)\Lambda_N(j)$ holds.*

Remark 2.1.1. If in Theorem 2.1 we take $\lambda_{n0} = 1 (\forall n \in \boldsymbol{N}_0)$ then the following holds:

(1.) $\Lambda_N(0) = 0$ $(\forall N \in \boldsymbol{N}_0)$ and $\Lambda_\infty(0) = \lim_{N\to\infty} \Lambda_N(0) = 0$.

(2.) $L_k(\chi_0,\Lambda,x) = \chi_0(x) = 1$ $(\forall k \in \boldsymbol{N}_0,\ \forall x \in G)$.

(3.) $E_N(\chi_0,\Lambda,x) = 0$ $(\forall N \in \boldsymbol{N}_0,\ \forall x \in G)$.

(4.) $[L_k(f,\Lambda,x)]^\wedge(\chi_0) \equiv \hat{L}_k(0) = \hat{f}(0)$ $(\forall k \in \boldsymbol{N}_0)$ and

(5.) $[E_N(f,\Lambda,x)]^\wedge(\chi_0) \equiv \hat{E}_N(0) = 0$ $(\forall N \in \boldsymbol{N})$.

Theorem 2.2. *Let G, f and Λ be as in Definition* 1.1. *Following statement hold:*

(i) *If $\lambda_{k0} = 1$ $(\forall k \in \boldsymbol{N}_0)$ and $f(x) = C$ $(\forall x \in G)$ (C-constant), then $f^\Lambda(x) = 0$ $(\forall x \in G)$.*

(ii) *If f and g are Λ-differentiable functions at a point $x \in G$, then the function $F := f + g$ is Λ-differentiable at x and $(f+g)^\Lambda(x) = f^\Lambda(x) + g^\Lambda(x)$.*

(iii) *If f is a Λ-differentiable function at a point $x \in G$ and C is a constant, then the function $\varphi := C \cdot f$ is Λ-differentiable at x and $(C f)^\Lambda(x) = C f^\Lambda(x)$.*

Theorem 2.3. *Let G, f and Λ be as in Definition* 1.1. *Let $L_k(f,\Lambda,x)$ be a continuous function at a point $x_0 \in G$ $(\forall k \in \boldsymbol{N}_0)$ (this condition is automatically fulfilled when Λ is a triangular matrix). If f is a Λ-differentiable function in some neighborhood of the point x_0, then*

$$\sigma(f,\Lambda,x) = \lim_{k\to\infty} L_{n_k}(f,\Lambda,x)$$

is a continuous function at x_0.

Theorem 2.4. *Let G, f and Λ be as in Definition* 1.1. *If $E_N(f,\Lambda,x) \in L^1(G)$ $(\forall N \in \boldsymbol{N})$ and $g(x)$ is a uniform Λ-derivative of a function f on G, then $g \in L^1(G)$ and $\hat{g}(j) = \hat{f}(j)\Lambda_\infty(j)$.*

Remark 2.1. Theorem 2.4 gives us a motivation to introduce sets $W(L^1,\Lambda)$ and $E(L^1,\Lambda)$ by formulas:

$$W \equiv W(L^1,\Lambda) := \{f \in L^1(G) : (\exists g \in L^1(G))\hat{g}(j) = \hat{f}(j)\Lambda_\infty(j),\ \forall j \in \boldsymbol{N}_0\}$$
$$E = E(L^1,\Lambda) := \{f \in L^1(G) : E_N(f,\Lambda) \in L^1(G),\ \forall N \in \boldsymbol{N}_0\}$$

Theorem 2.5. *Let G, f, Λ, $\mathcal{D}(\Lambda,G)$ and $\mathcal{UD}(\Lambda,G)$ be as in Definition 1.1. Then $E \cap \mathcal{UD} \subseteq E \cap W \subseteq \mathcal{D}$ where we used $\mathcal{UD}$ and $\mathcal{D}$ instead of $\mathcal{UD}(\Lambda,G)$ and $\mathcal{D}(\Lambda,G)$, respectively.*

3. Proofs

(3.1.) Prof of the theorem 2.1.

(i) Knowing that $\hat{\chi}_j(s) = \int\limits_G \chi_j(x)\overline{\chi}_s(x) = \begin{cases} 0, & s \neq j \\ 1, & s = j, \end{cases}$

one obtains $L_k(\chi_j,\Lambda,x) = \sum\limits_{s=0}^{\infty}\lambda_{ks}\hat{\chi}_j(s)\chi_s(x) = \lambda_{kj}\chi_j(x)$.

(ii) From (i) one obtains $\sigma(\chi_j,\Lambda,x) = \lim\limits_{k\to\infty} L_k(\chi_j,\Lambda,x) = \lim\limits_{k\to\infty}\lambda_{kj}\chi_j(x) = \lambda_{\infty j}\chi_j(x)$. Now we have

$$\begin{aligned} E_N(\chi_j,\Lambda,x) &= \sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}\left[\sigma(f,\Lambda,x) - L_i(f,\Lambda,x)\right] = \\ &= \sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}\left[\lambda_{\infty j}\chi_j(x) - \lambda_{ij}\chi_j(x)\right] = \\ &= \chi_j(x)\sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}(\lambda_{\infty j} - \lambda_{ij}) = \chi_j(x)\Lambda_N(j)\,. \end{aligned}$$

(iii) Follows from Definition 1.1 and (ii).

(iv) Knowing that the series $L_k(f,\Lambda,x) = \sum\limits_{s=0}^{\infty}\lambda_{ks}\hat{f}(s)\chi_s(x)$ is uniformly convergent on G (by the premises), one obtains that the series $L_k(f,\Lambda,x)\overline{\chi}_j(x)$ is uniformly convergent on G ($j \in \boldsymbol{N}_0$ is arbitrary).

From this one obtains

$$[L_k(f,\Lambda,x)]^{\wedge}(j) = \int_G \left[\sum_{s=0}^{\infty} \lambda_{ks}\hat{f}(s)\chi_s(x)\right] \overline{\chi}_j(x) =$$

$$= \sum_{s=0}^{\infty} \lambda_{ks}\hat{f}(s) \int_G \chi_s(x)\overline{\chi}_j(x) = \lambda_{kj}\hat{f}(j).$$

(v) As $L_k(f,\Lambda,x) \to \sigma(f,\Lambda,x)$ unifirmly on G, from (iv) one obtains

$$[E_N(f,\Lambda,x)]^{\wedge}(j) = \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} \left[\int_G \sigma(f,\Lambda,x)\overline{\chi}_j(x) - \int_G L_i(f,\Lambda,x)\overline{\chi}_j(x) = \right.$$

$$= \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} [\hat{\sigma}(f,\Lambda,x)(j) - \hat{L}_i(f,\Lambda,x)(j)] =$$

$$= \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} [\lambda_{\infty j}\hat{f}(j) - \lambda_{ij}\hat{f}(j)] = \hat{f}(j)] = \hat{f}(j)\Lambda_N(j).$$

This concludes the proof of Theorem 2.1

(3.2.) Prof of the theorem 2.2.

(i) As $\lambda_{k0} = 1$ $(\forall k \in \boldsymbol{N}_0)$, we have $\lambda_{\infty 0} = 1$. If C is a constant and $s \in \boldsymbol{N}_0$, we have

$$\hat{C}(s) = \int_G C\overline{\chi}_s = \begin{cases} C, & \text{for } s = 0 \\ 0, & \text{for } s \neq 0. \end{cases}$$

Now we have $L_k(C,\Lambda,x) = \sum_{s=0}^{\infty} \lambda_{ks}\hat{C}(s)\chi_s(x) = \lambda_{k0}C = 1 \cdot C = C$ and

$$\sigma(C,\Lambda,x) = \lim_{k\to\infty} L_k(C,\Lambda,x) = \lambda_{\infty 0}C = 1C = C.$$

From this one obtains

$$E_N(C,\Lambda,x) = \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} \left[\sigma(C,\Lambda,x) - L_i(C,\Lambda,x)\right] = 0, \quad (\forall N \in \boldsymbol{N}),$$

and finally $C^{\Lambda}(x) = \lim_{N\to\infty} E_N(C,\Lambda,x) = 0$.

(ii) If is sufficient to prove that each of the operators $L_k(f,\Lambda,x)$, $\sigma(f,\Lambda,x)$ and $E_N(f,\Lambda,x)$ is aditive in the first variable. From premisses we know that f and g are Λ-differentiable at $x \in G$, so we have

$$L_k(f,\Lambda,x) \to \sigma(f,\Lambda,x)\ (k\to\infty),\ \ L_k(g,\Lambda,x) \to \sigma(g,\Lambda,x)\ (k\to\infty).$$

and there exist Λ-derivatives

$$f^{\Lambda}(x) = \lim_{N\to\infty} E_N(f,\Lambda,x) \quad \text{and} \quad g^{\Lambda}(x) = \lim_{N\to\infty} E_N(g,\Lambda,x).$$

Now we have

$$\begin{aligned} L_k(f+g,\Lambda,x) &= \sum_{s=0}^{\infty} \lambda_{ks}(f+g)^{\wedge}(s)\chi_s(x) = \\ &= \sum_{s=0}^{\infty} \lambda_{ks}\big[\hat{f}(s)+\hat{g}(s)\big]\chi_s(x) = \\ &= \sum_{s=0}^{\infty} \lambda_{ks}\hat{f}(s)\chi_s(x) + \sum_{s=0}^{\infty} \lambda_{ks}\hat{g}(s)\chi_s(x) = \\ &= L_k(f,\Lambda,x) + L_k(g,\Lambda,x) \end{aligned}$$

(because series $\sum_{s=0}^{\infty}\lambda_{ks}\hat{f}(s)\chi_s(x)$ and $\sum_{s=0}^{\infty}\lambda_{ks}\hat{g}(s)\chi_s(x)$ anre convergent by assumption),

$$\begin{aligned} \sigma(f+g,\Lambda,x) &= \lim_{k\to\infty} L_k(f+g,\Lambda,x) = \lim_{k\to\infty}\big[L_k(f,\Lambda,x)+L_k(g,\Lambda,x)\big] = \\ &= \lim_{k\to\infty} L_k(f,\Lambda,x) + \lim_{k\to\infty} L_k(g,\Lambda,x) = \sigma(f,\Lambda,x)+\sigma(g,\Lambda,x) \end{aligned}$$

(because $\lim_{k\to\infty} L_k(f,\Lambda,x)$ and $\lim_{k\to\infty} L_k(f,\Lambda,x)$ exist by the premises), and

$$E_N(f+g,\Lambda,x) = \sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}\big[\sigma(f+g,\Lambda,x) - L_i(f+g,\Lambda,x)\big] =$$

$$= \sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}\big\{\big[\sigma(f,\Lambda,x)-L_i(f,\Lambda,x)\big]+\big[\sigma(g,\Lambda,x)-L_i(g,\Lambda,x)\big]\big\} =$$

$$= \sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}\big[\sigma(f,\Lambda,x)-L_i(f,\Lambda,x)\big] +$$

$$+ \sum_{k=-1}^{N-1}\sum_{i=m_k}^{m_{k+1}-1}\big[\sigma(g,\Lambda,x)-L_i(g,\Lambda,x)\big] =$$

$$= E_N(f,\Lambda,x) + E_N(g,\Lambda,x).$$

(iii) It is sufficient to prove that each of the operators $L_k(f,\Lambda,x)$, $\sigma(f,\Lambda,x)$ and $E_N(f,\Lambda,x)$ is homogenous in the first variable. This can be easily proved following the lines of the proof of the statement (ii). Theorem 2.2 is proved.

(3.3.) Prof of the theorem 2.3.

Let us chose arbitrary $\varepsilon > 0$. By the premises of the theorem, there exists a neighborhood of x_0, say $x_0 + G_s$, such that the function f is Λ-differentiable in that neighborhood. It means there exist finite limits:

$$f^\Lambda(x_0+t) = \lim_{N\to\infty} E_N(f,\Lambda,x_0+t) =$$

$$= \lim_{N\to\infty} \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} [\sigma(f,\Lambda,x_0+t) - L_i(f,\Lambda,x_0+t)] \quad (\forall t \in G_s) \text{ and}$$

$$f^\Lambda(x_0) = \lim_{N\to\infty} E_N(f,\Lambda,x_0) = \lim_{N\to\infty} \sum_{k=-1}^{N-1} \sum_{i=m_k}^{m_{k+1}-1} [\sigma(f,\Lambda,x_0) - L_i(f,\Lambda,x_0)] .$$

As series on the right sides of last two equalities are convergent, their members tend to zero as $k \to \infty$, and we have

$$\sigma(f,\Lambda,x_0+t) - L_i(f,\Lambda,x_0+t) \to 0 \quad (k\to\infty) \quad (\forall t \in G) \text{ and}$$

$$\sigma(f,\Lambda,x_0) - L_i(f,\Lambda,x_0) \to 0 \quad (k \to \infty) .$$

This means that there exist numbers $i_1 = i_1(\varepsilon)$ and $i_2 = i_2(\varepsilon)$ such that for every $i \geq i_3 = \max\{i_1, i_2\}$ holds

$$\begin{aligned} &|\sigma(f,\Lambda,x_0+t) - L_i(f,\Lambda,x_0+t)| < \frac{\varepsilon}{3} \quad (\forall t \in G_s) \quad \text{and} \\ &|\sigma(f,\Lambda,x_0) - L_i(f,\Lambda,x_0)| < \frac{\varepsilon}{3} . \end{aligned} \tag{3.3.1}$$

By the premises of the theorem, for every $i \in N_0$ the function $L_i(f,\Lambda,x)$ is continuous in x_0. Let us choose an $i_0 \geq i_3$. Then there exists a neighborhood of the point x_0, some $x_0 + G_{j(i_0)}$ such that for every $t \in G_{j(i_0)}$. holds

$$|L_{i_0}(f,\Lambda,x_0+t) - L_{i_0}(f,\Lambda,x_0)| < \frac{\varepsilon}{3} . \tag{3.3.2}$$

Without loss of generality we can assume $G_s \subseteq G_{j(i_0)}$. Now, from (3.3.1) and (3.3.2) we can conclude that for every $t \in G_s$

$$\begin{aligned}
&|\sigma(f,\Lambda,x_0+t)-\sigma(f,\Lambda,x_0) = \\
&= |\sigma(f,\Lambda,x_0+t)-L_{i_0}(f,\Lambda,x_0+t)+ \\
&+ L_{i_0}(f,\Lambda,x_0+t)-L_{i_0}(f,\Lambda,x_0)+L_{i_0}(f,\Lambda,x_0)-\sigma(f,\Lambda,x_0)| \le \\
&\le |\sigma(f,\Lambda,x_0+t)-L_{i_0}(f,\Lambda,x_0+t)|+|L_{i_0}(f,\Lambda,x_0+t)- \\
&- L_{i_0}(f,\Lambda,x_0)|+|L_{i_0}(f,\Lambda,x_0)-\sigma(f,\Lambda,x_0)| < \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\,.
\end{aligned}$$

So, the function $\sigma(f,\Lambda,x)$ is continuous at x_0.

(3.4.) Prof of the theorem 2.4.

From $E_N(f,\Lambda,x) \to g(x)$ $(N \to \infty)$ uniformly on G, one obtains $g \in L^1(G)$ (by the Lebesque dominated convergence theorem), and

$$\|E_N(f,\Lambda)-g\|_1 \to 0 \ (N \to \infty), \ \text{ so certainly } \ |\hat{E}_N(f,\Lambda)-\hat{g}| \to 0 \ (N \to \infty)$$

This means that

$$[E_N(f,\Lambda)]^\wedge(j) \to \hat{g}(j)\,, \ (N \to \infty) \ (\forall j \in \boldsymbol{N}_0)\,.$$

By the statement (v) of Theorem 2.1, for every $j \in \boldsymbol{N}_0$ we have

$$[E_N(f,\Lambda)]^\wedge(j) \to \hat{f}(j)\Lambda_\infty(j)\,.$$

From the last two relations one obtains

$$\hat{g}(j)=\hat{f}(j)\Lambda_\infty(j) \quad (\forall j \in \boldsymbol{N}_0)\,.$$

Theorem 2.4 is proved.

(3.5.) Prof of the theorem 2.5.

a) Let us take $f \in E \cap \mathcal{UD}$. This means that

$$E_N(f,\Lambda) \in L^1(G) \ \forall N \in \boldsymbol{N}_0) \ \text{ and } \ \lim_{N\to\infty} E_N(f,\Lambda,x)=f^\Lambda(x)=:g(x)$$

uniformly on G. By Theorem 2.4

$$g \in L^1(G) \ \text{ and } \ \hat{g}(j)=\hat{f}(j)\Lambda_\infty(j) \ (\forall j \in \boldsymbol{N}_0).$$

From this and the definition of the set $W = W(L^1, \Lambda)$ follows that $f \in W$.

b) Let us take $f \in W \cap E$. Then there exsists $g \in L^1(G)$ such that for every $j \in \boldsymbol{N}_0$, $\hat{g}(j) = \hat{f}(j)\Lambda_\infty(j)$. By the statement (v) of Theorem 2.1 we have

$$[E_N(f, \Lambda)]^\wedge(j) = \hat{E}_N(j) = \hat{f}(j)\Lambda_N(j) \quad (\forall j \in \boldsymbol{N}_0).$$

From the last equality we have

$$\hat{E}_N(j) = \hat{g}(j)\frac{\Lambda_N(j)}{\Lambda_\infty(j)},$$

and this implies

$$\hat{E}_N(j) - \hat{g}(j) = \hat{g}(j)\left[\frac{\Lambda_N(j)}{\Lambda_\infty(j)} - 1\right] \to 0 \ (N \to \infty), \ (\forall j \in \boldsymbol{N}_0).$$

From the uniqueness theorem, one obtains $\lim_{N\to\infty}[E_N(x) - g(x)] = 0$ a.e. on G, i.e. $\lim_{N\to\infty} E_N(x) = g(x)$ a.e. on G. This, by the definition of the Λ-derivative means that $f^\Lambda(x) = g(x)$ a.e. on G, i.e. $f \in \mathcal{D} = \mathcal{D}(\Lambda, G)$. This implies that $E \cap W \subseteq \mathcal{D}$. Theorem 2.5 is proved.

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ДИФЕРЕНЦИРАЊЕ НА ФУНКЦИИ НА ГРУПИ НА ВИЛЕНКИН

Медо Пепиќ

Р е з и м е

Во овој труд го разгледуваме концептот на диференцирање на комплексни функции дефинирани на групи на Виленкин, од аспект на сумабилности на редови на Виленкин. Во таа смисла даваме матрична интерпретација на C. N. Onnewer-мот концепт на диференцирање [5, Дефиниција 3], и наша генерализација на тој концепт. Исто така ги даваме својствата на овој генерализиран извод. Понатаму, се покажува дека секој метод на сумирање на редови на Виленкин, на некој начин, дефинира некој концепт на диференцирање на интеграбилни функции на групите на Виленкин.

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