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SIMILARITY AND QUASISIMILARITY OF BILATERAL OPERATOR VALUED WEIGHTED SHIFTS

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Abstract. In this paper we study the problem of quasisimilarity of bilateral operator valued weighted shifts which is a generalization of a result of Fialkow for the scalar valued weighted shifts.

Let H be a complex Hilbert space and let $l^2(H)$ be the Hilbert space of all sequences $(x_n)_{n=-\infty}^{+\infty}$ such that $\sum_{n=-\infty}^{\infty} ||x_n||^2<\infty$, $x_n \in H$, with scalar product $((x_n),(y_n)) = \sum_{n=-\infty}^{\infty} (x_n,y_n)$.

When convinient we will write $(x_n) = \sum_{n=-\infty}^{\infty} \oplus x_n$ or

$$(x_n) = (\ldots,x_{-2},x_{-1},\boxed{x_0},x_1,x_2,\ldots)$$

where the square denotes the zero position.

Let $(A_i)_{i=-\infty}^{+\infty}$ be a uniformly bounded sequence of positive invertible operators, then the operator A on $l^2(H)$ defined by

$$A(\ldots,f_{-1},f_0,f_1,f_2,) =$$
$$= (\ldots,A_{-2},f_{-2},\boxed{A_{-1}f_{-1}},A_0f_0,\ldots)$$

is called a bilateral operator valued weighted shift with weights $(A_i)_{i=-\infty}^{+\infty}$.

Again we make a remark that the square meanns the zero position.

Without loss of generality we will assume that the operator weights A_i are positive, since each invertible operator valued weighted shift is unitarily equivalent to the operator weighted shift with positive weights, see Lambert [6].

Let B(H) be the algebra of all bounded, linear operators on H. An operator X in B(H) is quasi-invertible if X is injective and has a dense range (i.e. Ker(X)=Ker(X*)={0}).

Operators A and B in B(H) are quasisimilar if there exist operators X and Y wixh are quasi-invertible such that AX=XB and YA=BY.

It is clear that similarity if two operators implies quasi-similarity.

The question of similarity and quasisimilarity of scalar weighted shifts are studied by several authors, Kelly (see Halmos [1]), Hoover [2], Fialkow [3], [4], [5], and Williams [7] and others.

In this note we will generalize the result of Fialkow [3] to the case of bilateral operator valued weighted shifts.

Theorem 1. Let A and B are operator valued weighted shifts with weights $\{A_i\}_{i=-\infty}^{+\infty}$ and $\{B_i\}_{i=-\infty}^{+\infty}$ respectively, and suppose that there exists an integer k such that

(a) $\sup_{i\geq\max(1-k,1)} \{||A_{i-1+k}\ldots A_0 B_0^{-1} B_1^{-1}\ldots B_{i-1}^{-1}||\} < \infty$

and

(b) $\sup_{i\geq\max(1-k,1)} \{||A_{-i}^{-1} A_{-(i-1)}^{-1}\ldots A_{-1}^{-1}\cdot B_{-1} B_{-2}\ldots B_{-(i+k)}||\} < \infty$

then there exists a quasiinvertible operator X such that AX=XB.

Proof. We will find a diagonal operator D and a product $U^k D$ will be required solution. The operator U is a bilateral un-weighted shift which is defined as follows

$$U(\ ,\ldots,f_{-1},\boxed{f_0},f_1,\ldots\ ) = (\ldots f_{-2},\boxed{f_{-1}},f_0\ldots\ )$$

For the definitions of the diagonal elements D_i of operator D the following case we will consider:

Case 1. If $k \geq 2$ we set

$$\text{For } i \geq 1,\ D_i = A_{i-1+k}\ldots A_0 B_0^{-1}\cdot B_1^{-1}\ldots B_{i-1}^{-1} \tag{1}$$

$$D_k = A_{k-1}\ldots A_1 A_0 \tag{2}$$

$$\text{For } -k+1 \leq i \leq -1 \tag{3}$$

$$D_i = B_i\ldots B_{-1} A_0\ldots A_{k+1-i}$$

$$D_{-k} = B_{-k}\ldots B_{-1} \tag{4}$$

For $i \geq 1$, we set (5)

$$D_{-(i+k)} = A_{-i}^{-1}A_{-(i-1)}^{-1}\dots A_{-1}^{-1}B_{-1}B_{-2}\dots B_{-(k+i)}$$

Case 2. If k=1 equation (3) may be deleted.

Case 3. If k=0, operators (2)-(4) may be replaced by the operator

$$D_o = I$$

Case 4. If $k \leq -2$ $(1-k \geq 3)$.

For $i \geq 1-k$ we set (1)

$$D_i = A_{i-1+k}\dots A_1A_oB_o^{-1}B_1^{-1}\dots B_{i-1}^{-1}$$

$$D_{-k} = B_o^{-1}\dots B_{-k-1}^{-1} \qquad (2)$$

For $1 \leq i \leq -(k+1)$ we set (3)

$$D_{-k-i} = A_{-i}^{-1}\dots A_{-1}^{-1}B_oB_1\dots B_{-k-i-1}$$

$$D_o = A_k^{-1}\dots A_o^{-1} \qquad (4)$$

For $i \geq 1-k$ we set (5)

$$D_{-i-k} = B_{-(1+k)}\dots B_{-1}A_{-1}^{-1}A_{-2}^{-1}\dots A_{-i}^{-1}$$

The conditions (a) and (b) imply that the operator X can be extended to a quasi-invertible operator X in $B(l^2(H))$. We will show that the equation AX=XB holds.

We consider Case 1, and $i \geq 1$. Let $\hat{f}_i=(\dots,\boxed{0},\dots,0,f_i,00)$. Then we have

$$B\hat{f}_i = (\dots,\boxed{0},0,\dots,0,B_if_i,0,\dots)$$

where B_if_i is a vector on i+1 position. If we look at the projection onto i+k+1 coordinate space we get

$$P_{i+k+1}XB\hat{f}_i = A_{i+k}A_{i-1+k}\dots A_oB_o^{-1}B_1^{-1}\dots B_{i-1}^{-1}B_i^{-1}B_if_i =$$
$$= A_{i+k}\dots A_oB_o^{-1}B_1^{-1}\dots B_{i-1}^{-1}f_i$$

On the other hand we have

$$P_{i+k+1}AX\hat{f}_i = P_{i+k+1}A(\dots,\boxed{0},\dots,0,D_if_i,\dots),$$

where D_if_i is a vector onto k+i position. So we have

$$P_{i+k+1}AX\hat{f}_i = A_{i+k}(A_{i-1+k}\ldots A_1A_0B_0^{-1}B_1^{-1} \ldots B_{i-1}^{-1}f_i)$$

therefore we have proved that

$$P_{i+k+1}AX\hat{f}_i = P_{i+k+1}XB\hat{f}_i$$

The other projections are zero, so it is shown that

$$AX\hat{f}_i = XB\hat{f}_i$$

The set of a linear combinations of vectors of the form $\hat{f}_i$ are dense in the space $l^2(H)$ so the equation AX=XB holds on the whole space $l^2(H)$.

Theorem 2. If A and B are operator valued weighted shifts with weights $\{A_i\}_{i=-\infty}^{+\infty}$ and $\{B_i\}_{i=-\infty}^{+\infty}$ respectively, and suppose that the following conditions hold

(a) There is an integer k such that

$$\sup_{i\geq\max(1-k,1)} \{||A_{i-1+k}\ldots A_0B_0^{-1}B_1^{-1}\ldots B_{i-1}^{-1}||\} < \infty$$

and

$$\sup_{i\geq\max(1-k,1)} \{||B_{-(i+k)}\ldots B_{-1}A_{-1}^{-1}A_{-2}^{-1}\ldots A_{-i}^{-1}||\} < \infty$$

(b) There exists an integer m such that

$$\sup_{i\geq\max(1-k,1)} \{||B_{i-1+m}\ldots B_0A_0^{-1}A_1^{-1}\ldots A_{i-1}^{-1}||\} < \infty$$

and

$$\sup_{i\geq\max(1-k,1)} \{||A_{-(i+m)}\ldots A_{-1}B_{-1}^{-1}B_{-2}^{-1}\ldots B_{-i}^{-1}||\}$$

Then operators A and B are quasisimilar.

Proof. Condition (a) implies the existence of quasi-invertible operator X on the space $l^2(H)$ such that AX=XB.

Condition (b) imolies that there exist a quasi-invertible operator Y such that YA=BY.

R E F E R E N C E S

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СЛИЧНОСТ И КВАЗИСЛИЧНОСТ НА ДВОСТРАНИ ОПЕРАТОРСКО ТЕЖИНСКИ ШИФТОВИ

Новак Ивановски

Р е з и м е

Во оваа работа се дава критериум за квазисличност на два инвертибилни операторско тежински шифтови со позитивни тежини, што претставува обопштување на резултатот на Фјалкоу во случај на двострани скаларни шифтови.

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