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A STATEMENT OF DIFFERENTIAL CALCULUS IN CONTEXT OF CODOMAIN OF THE FUNCTION

JOVAN V. MALEŠEVIĆ

Abstract. In this article we give an analogue of Statement 3 from [1] with appropriate geometrical interpretation in sense of codomain (image) of the function $f(x)$.

Using theorem of Darboux, some generalizations and analogizes of theorems D. Trahan [2] and T. Flett [4] by Theorem 3 from [1] are given. In this article using Lemma 2 [2] of D. Trahan and Corollary 3 from [3], the following Statement, with the mentioned emphasis on the geometrical interpretation, is given.

Statement. *Let $f : [a,b] \longrightarrow R$ be differentiable function. If there exists a point $x_0 \in (a,b)$, such that $f''(x_0)$ exists and*

$$(1)\quad \Big(f'(r)-\frac{f(r)-f(x_0)}{r-x_0}\Big)\Big(\frac{f(b)-f(a)}{b-a}-\frac{f(r)-f(x_0)}{r-x_0}\Big)>0,\ \textit{for}\ r=a\ \textit{and}\ r=b,$$

then exist at least two values $\xi \in (a,b)$ such that

$$(2)\qquad f'(\xi_i)=\frac{f(\xi_i)-f(x_0)}{\xi_i-x_0},\quad i=1,2.$$

Proof. Let us introduce function

$$(3)\qquad F(x)=\begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} & : \ x\neq x_0\in(a,b)\\ f'(x_0) & : \ x=x_0 \end{cases}$$

It is true, for $x \neq x_0$:

$$F'(x)=\frac{f'(x)(x-x_0)-\big(f(x)-f(x_0)\big)}{(x-x_0)^2}$$

and

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{2(x-x_0)}$$
$$= \lim_{x\to x_0}\frac{f''(x)}{2} = \frac{f''(x_0)}{2},$$

ie. function $F(x)$ is differentiable over segment $[a,b]$ and $F'(x_0)=\dfrac{f''(x_0)}{2}$.

$\mathbf{1^0}$. Based on the sign of the difference $\frac{f(b)-f(a)}{b-a}-\frac{f(r)-f(x_0)}{r-x_0} \underset{(<)}{>} 0$, in $r=a$, we obtained one upper (lower) bound for value $F(a)$ as follows

$$F(a)=\frac{f(a)-f(x_0)}{a-x_0} \underset{(>)}{<} \frac{f(b)-f(a)}{b-a}. \tag{4}$$

Previous inequality is equivalent with

$$\big(f(b)-f(a)\big)(a-b+b-x_0) \underset{(>)}{<} \big(f(a)-f(b)+f(b)-f(x_0)\big)(b-a),$$

ie.

$$\big(f(b)-f(a)\big)(b-x_0) \underset{(>)}{<} \big(f(b)-f(x_0)\big)(b-a).$$

meaning that

$$\frac{f(b)-f(a)}{b-a} \underset{(>)}{<} \frac{f(b)-f(x_0)}{b-x_0} = F(b);$$

and finally

$$F(a)=\frac{f(a)-f(x_0)}{a-x_0} \underset{(>)}{<} \frac{f(b)-f(a)}{b-a} \underset{(>)}{<} \frac{f(b)-f(x_0)}{b-x_0} = F(b). \tag{5}$$

Therefore

$$F(a) \underset{(>)}{<} F(b) \iff \big(F(b)-F(a)\big) \underset{(<)}{>} 0. \tag{6}$$

$\mathbf{2^0}$. Based on the sign of the difference $\frac{f(b)-f(a)}{b-a}-\frac{f(r)-f(x_0)}{r-x_0} \underset{(<)}{>} 0$, in $r=b$, we obtained one lower (upper) bound for value $F(b)$ as follows

$$\frac{f(b)-f(a)}{b-a} \underset{(<)}{>} \frac{f(b)-f(x_0)}{b-x_0} = F(b), \tag{7}$$

Previous inequality is equivalent with

$$\big(f(b)-f(a)\big)(b-a+a-x_0) \underset{(<)}{>} \big(f(b)-f(a)+f(a)-f(x_0)\big)(b-a),$$

ie.

$$\big(f(b)-f(a)\big)(x_0-a)\underset{(>)}{<}\big(f(x_0)-f(a)\big)(b-a).$$

meaning that

$$F(a)=\frac{f(b)-f(a)}{b-a}\underset{(>)}{<}\frac{f(b)-f(x_0)}{b-x_0}=F(b);$$

and finally

$$F(a)=\frac{f(a)-f(x_0)}{a-x_0}\underset{(<)}{>}\frac{f(b)-f(a)}{b-a}\underset{(<)}{>}\frac{f(b)-f(x_0)}{b-x_0}=F(b).$$

Therefore

$$F(a)\underset{(<)}{>}F(b)\iff\big(F(b)-F(a)\big)\underset{(>)}{<}0. \tag{8}$$

3^0. Based on the sign of the difference $f'(r)-\frac{f(r)-f(x_0)}{r-x_0} \underset{(<)}{>} 0$, in $r=a$, we obtained

$$F'(a) = \frac{f'(x)(x-x_0)-\big(f(x)-f(x_0)\big)}{(x-x_0)^2}\Big|_{x=a}$$
$$= \frac{f'(a)(a-x_0)-\big(f(a)-f(x_0)\big)}{(a-x_0)^2}$$
$$= \frac{1}{a-x_0}\Big(f'(a)-\frac{f(a)-f(x_0)}{a-x_0}\Big) \underset{(>)}{<} 0,$$

ie.

$$F'(a) \underset{(>)}{<} 0. \tag{9}$$

Analogously, in $r=b$, we obtained

$$F'(b) = \frac{1}{b-x_0}\Big(f'(b)-\frac{f(b)-f(x_0)}{b-x_0}\Big) \underset{(<)}{>} 0,$$

ie.

$$F'(b) \underset{(<)}{>} 0. \tag{10}$$

4^0. From (6) and (9) follows

$$F'(a)\big(F(b)-F(a)\big) < 0, \tag{11}$$

and analogously, based on (8) and (10), we can conclude

$$F'(b)\big(F(b)-F(a)\big) < 0. \tag{12}$$

Therefore, is true

$$F'(a)\big(F(b)-F(a)\big) < 0 \;\;\wedge\;\; F'(b)\big(F(b)-F(a)\big) < 0. \tag{13}$$

Next, using Lemma 2 [2] and Corollary 3 [3] respectively, existence least two $\xi \in (a,b)$ follows such that $F'(\xi)=0$ and therefore follows (in Fletts denotation):

$$\big(\exists \xi_i \in (a,b)\big) f'(\xi_i) = \frac{f(\xi_i)-f(x_0)}{\xi_i - x_0}, \quad i=1,2. \tag{14}$$

Geometrically interpretation. In this part we geometrically determined part of codomain of the function f over segment $[a,b]$ in case of tangents in points A and B and their intersection $\big(f'(a)\neq f'(b)\big)$, based on the following conditions:

$$f'(a)-\frac{f(x_0)-f(a)}{x_0-a} > 0 \quad \wedge \quad f'(b)-\frac{f(b)-f(x_0)}{b-x_0} > 0. \tag{15}$$

¿From previous proof of the Statement we have:

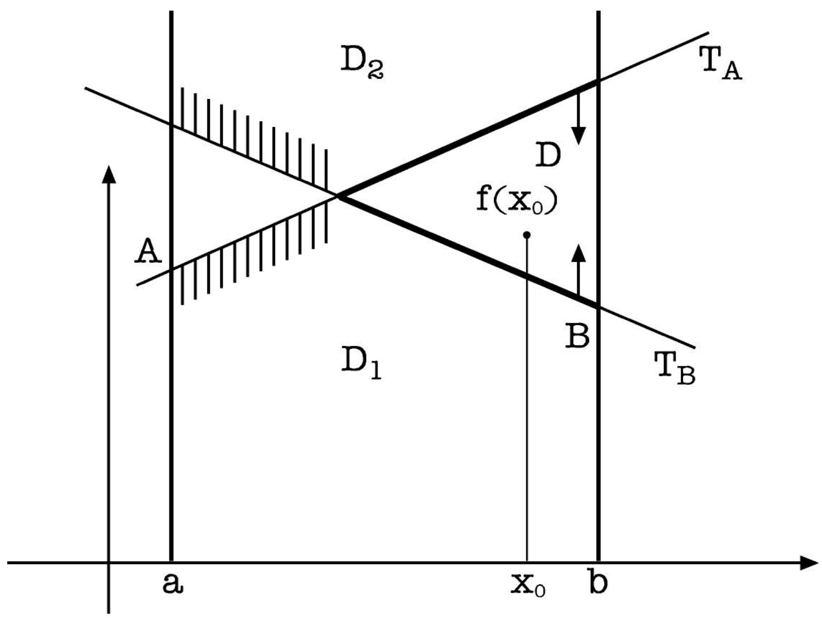
$$\frac{f(x_0)-f(a)}{x_0-a} < f'(a) \quad \wedge \quad f'(b) > \frac{f(b)-f(x_0)}{b-x_0},$$

ie.

(16)

$$f(x_0) < T_A : f(x_0) < f(a) + f'(a)(x_0 - a) \wedge f(x_0) > T_B : f(x_0) > f(b) + f'(b)(x_0 - b);$$

therefore we have the following Figure of function $f(x)$ over segment $[a, b]$ and tangent in point A (bellow this tangent is region D_1), and tangent in point B (above this tangent is region D_2). On this way in Figure we have interior region $D = D_1 \cap D_2$ for both possibility in (14). Special consideration in connection of codomain of the function $f(x)$ in regions D_1 and D_2 correspond relations (11) and (12) respectively.



Figure

REFERENCES

[1.] J.V. Malešević: *Jedna generalizacija formule Tejlora*, Matematički vesnik, 12 (27), 1975, 375-384.

[2.] D.H. Trahan: *A new type of mean values*, Mat. Magazine 1966. 39, No5, 264-266.

[3.] J.V. Malešević: *O jednoj teoremi G. Darboux-a i teoremi D. Trahan-a*, Matematički vesnik, 4 (17), 1980, 32.

[4.] T.H. Flett: *A mean value theorem*, Mat. Gazette, 1958, 42, 38-39.

ЕДНО ТВРДЕЊЕ ЗА ДИФЕРЕНЦИЈАЛНОТО СМЕТАЊЕ ВО КОНТЕКСТ НА КОДОМЕНОТ НА ФУНКЦИЈАТА

Јован Малешевиќ

Р е з и м е

Во овој труд даваме аналогно тврдење на тврдењето 3 од [1] со соодветна геометриска интерпретација во смисла на кодоменот (сликата) на функцијата $f(x)$.

Futoška 60, Novi Sad, Serbia
E-mail address: malesh@eunet.rs