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A NOTE ON HOMEOMORPHISMS

ZBIGNIEW DUSZYŃSKI

Abstract. Two topologies τ_w and τ'_w determined by a given topology τ on X are considered. Relation between homeomorphisms on (X,τ) and homeomorphisms on (X,τ_w) or (X,τ'_w) are indicated.

1. Preliminaries

Throughout the present paper (X,τ), (Y,σ) mean topological spaces on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset S in (X,τ) are denoted by $\mathrm{cl}\,(S)$ and $\mathrm{int}\,(S)$ respectively. A subset S of (X,τ) is said to be ***semi-open*** [6] (resp. ***semi-closed*** [1, Theorem 1.1]) if there exists an open set O with $O \subset S \subset \mathrm{cl}\,(O)$ (resp. if there exists a closed F with $\mathrm{int}\,(F) \subset S \subset F$). The family of all semi-open (resp. semi-closed) subsets of (X,τ) is denoted as $\mathrm{SO}\,(X,\tau)$ (resp. $\mathrm{SC}\,(X,\tau)$). In [6, Theorem 7] Levine proved that if $A \in \mathrm{SO}\,(X,\tau)$, then $A = G \cup N$ for a certain $G \in \tau$ and a certain nowhere dense N. Dlaska et al. made a deeper remark [3, Sec.1, p.1163]: $A \in \mathrm{SO}\,(X,\tau)$ if and only if $A = G_A \cup N_A$ with G_A being a suitable open set and a nowhere dense $N_A \subset \mathrm{Fr}\,(G_A)$ ($\mathrm{Fr}\,(S)$ stands for the boundary of S).

The remark of Dlaska et al. [3] concerning representation of semi-open sets can be reformulated as follows.

Lemma 1. [5]. *Let (X,τ) be a topological space. Then, $A \in \mathrm{SO}\,(X,\tau)$ if and only if $A = \mathrm{int}\,(A) \cup N$ for a certain $N \subset \mathrm{Fr}\,(\mathrm{int}\,(A))$.*

The reader is advised to compare the following lemma to Lemma 1.

Lemma 2. [5]. *For any space (X,τ), $B \in \mathrm{SC}\,(X,\tau)$ if and only if there exist $F \in \mathrm{c}(\tau)$ and $M \subset X$ with*

(1) $B = \mathrm{int}\,(F) \cup M$ *and*
(2) $M \subset \mathrm{Fr}\,(F)$.

2. Homeomorphisms

Lemma 3. *Let (X,τ) be any topological space. Let $\hat{\tau}_w$ denote the family of all subsets of X of the form $X \setminus \left(N \cup \bigcap_{\alpha\in A} G_\alpha\right)$, where A is arbitrary, $G_\alpha \in \tau$ for*

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each $\alpha \in A$, and N is nowhere dense in (X, τ). Then $\hat{\tau}_w$ is a basis for a certain topology, designed as τ_w, on X.

Proof. One easily checks that $\emptyset, X \in \hat{\tau}_w$. Consider arbitrary $V_1 = X \setminus \left(N_1 \cup \bigcap_{\alpha \in A_1} G_\alpha\right) \in \hat{\tau}_w$ and $V_2 = X \setminus \left(N_2 \cup \bigcap_{\beta \in A_2} G_\beta\right) \in \hat{\tau}_w$. We have (use [4, Theorem 4.2(1)])

$$V_1 \cap V_2 = X \setminus \left[\left(N_1 \cup \bigcap_{\alpha \in A_1} G_\alpha\right) \cup \left(N_2 \cup \bigcap_{\beta \in A_2} G_\beta\right)\right] =$$

$$= X \setminus \left[(N_1 \cup N_2) \cup \bigcap_{(\alpha,\beta) \in A_1 \times A_2} (G_\alpha \cup G_\beta)\right].$$

Thus $V_1 \cap V_2 \in \hat{\tau}_w$. □

Now, we offer a few results related to the just introduced topology τ_w. Recall that a function $f \colon (X, \tau) \to (Y, \sigma)$ is said to be ***irresolute*** [2] if $f^{-1}(U) \in \mathrm{SO}\,(X, \tau)$ for each $U \in \mathrm{SO}\,(Y, \sigma)$.

Lemma 4. *Let (X, τ) and (Y, σ) be any topological spaces. If $f \colon (X, \tau) \to (Y, \sigma)$ is irresolute, then $\hat{f} \colon (X, \tau_w) \to (Y, \sigma_w)$ is continuous, where $\hat{f}(x) = f(x)$ for each $x \in X$.*

Proof. Let $T = Y \setminus \left(N \cup \bigcap_{\alpha \in A} G_\alpha\right)$ be any member of $\hat{\sigma}_w$, where N is nowhere dense in (Y, σ) and $G_\alpha \in \sigma$ for each $\alpha \in A$. Then

$$\hat{f}^{-1}(T) = X \setminus \left(\hat{f}^{-1}(N) \cup \bigcap_{\alpha \in A} \hat{f}^{-1}(G_\alpha)\right),$$

where $\hat{f}^{-1}(N) \in \mathrm{SC}\,(X, \tau)$ and $\hat{f}^{-1}(G_\alpha) \in \mathrm{SO}\,(X, \tau)$ for each $\alpha \in A$, because f is irresolute. Obviously, for certain nowhere dense (in (X, τ)) M and M_α, and $O, O_\alpha \in \tau$ ($\alpha \in A$) we have

$$\hat{f}^{-1}(T) = X \setminus \left((O \cup M) \cup \bigcap_{\alpha \in A} (O_\alpha \cup M_\alpha)\right) = X \setminus \left[\left(M \cup \bigcap_{\alpha \in A} M_\alpha\right) \cup \bigcap_{\alpha \in A} (O \cup O_\alpha)\right].$$

This shows that $f^{-1}(T) \in \hat{\tau}_w$. Thus the proof is complete. □

A function $f \colon (X, \tau) \to (Y, \sigma)$ is ***pre-semi-open*** [2] if $f(V) \in \mathrm{SO}\,(Y, \sigma)$ for each $V \in \mathrm{SO}\,(X, \tau)$.

Lemma 5. *Let (X, τ) and (Y, σ) be any topological spaces. If $f \colon (X, \tau) \to (Y, \sigma)$ is a pre-semi-open bijection, then the function $\hat{f} \colon (X, \tau_w) \to (Y, \sigma_w)$, defined as in Lemma 4, is open.*

Proof. By hypothesis, the inverse function $f^{-1} \colon (Y, \sigma) \to (X, \tau)$ is irresolute, So, by Lemma 4, $\widehat{\left(f^{-1}\right)} \colon (Y, \sigma_w) \to (X, \tau_w)$ is continuous. Therefore, $\hat{f} \colon (X, \tau_w) \to (Y, \sigma_w)$ is open. □

A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be a ***semi-homeomorphism*** [2] if it is irresolute, pre-semi-open, and bijective.

Theorem 1. *Let (X,τ) and (Y,σ) be arbitrary and let $f\colon (X,\tau) \to (Y,\sigma)$ be a function. We have the following sequence of implications: f is a homeomorphism $\overset{(1)}{\Longrightarrow}$ f is an open semi-homeomorphism $\overset{(2)}{\Longrightarrow}$ f is a semi-homeomorphism $\overset{(3)}{\Longrightarrow}$ $\hat{f}\colon (X,\tau_w) \to (Y,\sigma_w)$ is a homeomorphism, where none of them need not be reversible.*

Proof. (1) is clear by [2, Theorem 1.9], while (3) follows from Lemmas 4 and 5. For non-reversibility of (1) (resp. (2); (3)) we refer to Example 1 below (resp. [2, Example 1.1], Example 2 below). □

Example 1. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{a,b\}\}$. The identity $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is an open semi-homeomorphism, but it is not a homeomorphism.*

Example 2. *Let $X = \{a,b\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. Then, the identity $\hat{\mathrm{id}}\colon (X,\tau_w) \to (X,\sigma_w)$ is a homeomorphism, but $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is not open.*

Remark that openness and semi-homeomorphity are independent notions.

Example 3. (**a**). *Let $X=\{a,b,c\}$, $\tau=\{\emptyset, X, \{a\}\}$ and $\sigma=\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. The identity $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is open and it is not a semi-homeomorphism.*
(**b**). [2, Example 1.1] *guarantees the existence of a semi-homeomorphism which is not open.*

Remark 1. Note that implications (1) and (2) in Theorem 1 may be substituted by the following: f is a homeomorphism $\overset{(1')}{\Longrightarrow}$ f is an continuous semi-homeomorphism $\overset{(2')}{\Longrightarrow}$ f is a semi-homeomorphism. The converses of $(1')$ and $(2')$ may not be true, as seen by respective examples below.

Example 4. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. Then $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is a continuous semi-homeomorphism, but it is not a homeomorphism, since $\mathrm{id}(\{a,b\}) \notin \sigma$.*

Example 5. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{b\}, \{b,c\}\}$, $\sigma = \{\emptyset, Y, \{b\}, \{a,b\}\}$. The identity $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is a semi-homeomorphism, but it is not continuous.*

Continuity and semi-homeomorphity are independent of each other. Because of Example 5, it is enough to recommend the following one.

Example 6. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$ and $\sigma = \{\emptyset, X, \{a\}\}$. Then $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is continuous while it is not a semi-homeomorphism* $(\mathrm{id}(\{b,c\}) \notin \mathrm{SO}(X,\sigma))$.

With the following lemma, the topology τ'_w was introduced in [5].

Lemma 6. *Let (X,τ) be any topological space. Let $\hat{\tau}'_w$ denote the family of all subsets of X of the form $X \setminus \bigcap_{\alpha\in A}(G_\alpha \cup N_\alpha)$, where A is arbitrary, $G_\alpha \in \tau$, and N_α is nowhere dense in (X,τ) for each $\alpha \in A$. Then $\hat{\tau}'_w$ is a basis for a certain topology, designed as τ'_w, on X.*

Lemmas 7&8 below can be obtained similarly as Lemmas 4&5 respectively.

Lemma 7. *Let* (X,τ) *and* (Y,σ) *be any topological spaces. If* $f:(X,\tau)\to(Y,\sigma)$ *is irresolute, then* $\hat{f}':(X,\tau'_w)\to(Y,\sigma'_w)$ *is continuous, where* $\hat{f}'(x)=f(x)$ *for each* $x\in X$.

Lemma 8. *Let* (X,τ) *and* (Y,σ) *be any topological spaces. If* $f:(X,\tau)\to(Y,\sigma)$ *is a pre-semi-open bijection, then the function* $\hat{f}':(X,\tau'_w)\to(Y,\sigma'_w)$ *obtained as in Lemma* 7*, is open.*

Theorem 2. *Let* (X,τ) *and* (Y,σ) *be arbitrary and let* $f\colon(X,\tau)\to(Y,\sigma)$ *be a function. If* f *is a homeomorphism, then* $\hat{f}'\colon(X,\tau'_w)\to(Y,\sigma'_w)$ *is a homeomorphism.*

Proof. Lemmas 7&8. □

Example 2 shows that the implication in Theorem 2 may be irreversible.

Problem. Indicate a space (X,τ) such that the topologies τ_w and τ'_w considered above, are different.

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Casimirus the Great University
Institute of Mathematics
pl. Weyssenhoffa 11
85-072 Bydgoszcz
Poland