Математички Билтен
Книга 3—4 (XXIX—XXX) 1979—1980 (27—33)
Скопје — Југославија

TRANSFORM WHICH MAPS DERIVATIVES INTO GENERALIZED DIFFERENCES

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1. INTRODUCTION

In the papers [1], [2] and [3] D. S. Mitrinović investigated the numbers $\overline{R}_i^n$ defined by

$$\prod_{i=0}^{n-1} (x-(a+bi)) = \sum_{i=0}^{n} \overline{R}_i^n x^i, \ (a, \ b \in R).$$

In the paper [4] using the numbers $R_i^n = (-1)^{i-n} \overline{R}_i^n$ we defined a linear transform which maps the set of real functions having continuous derivatives of all orders into the set of all real sequences with the property that derivatives are mapped into differences defined by $Dv_n = v_{n+1} - (a+bn) v_n$, $D^k v_n = D(D^{k-1} v_n)$ $(a,b \in R, \ k \in N_0)$. We use the notations introduced in [4].

In this paper we will consider a linear transform which maps the set $C^\infty (R)$ of functions $f: R^2 \to R$ having continuous partial derivatives with respect to one variable into the set of real sequences $\{(v_n (y))\}$ with the property that partial derivatives are mapped into differences D. Further, we will consider a linear transform which maps the set $C^\infty (R \times R)$ of functions $f: R^2 \to R$ having continuous partial derivatives of all orders with respect to both variables into the set of real sequences $\{(v_{m,n})\}$ with the property that partial derivatives are mapped into partial diferences

$$D_m v_{m,n} = v_{m+1} - (a+bm) v_m, \ D_m^k = D_m (D_m^{k-1} v_{m,n}) \text{ and } D_n v_{m,n} =$$
$$= v_{m,n+1} - (a+bn) v_{m,n} \ D_n^k v_{m,n} = D_n (D_n^{k-1} v_{m,n}).$$

Also, we will apply these transforms and their inverse transfoms to solve some differential-difference and partial difference equations by establishing analogies between these equations and corresponding differential equations.

We list some basic properties of the numbers R_i^n which shall be of use later:

1^0 $R_i^n = 0$ for i 0, i n, $R_n^n = 1$

$$2^0 \quad R_0^n = \prod_{i=0}^{n-1} (a + bi), \quad R_0^0 = 1 \qquad (i,\ n \in N_0)$$

$$3^0 \quad R_i^{n+1} = R_{i+1}^n + (a + bn) R_i^n$$

2. BASIC DEFINITION AND PROPERTIES OF V AND R-TRANFSFORMS

DEFINITION 1. V-transform of a function $f(x, y) \in C^\infty(R)$ *is the sequence* $(v_n(y))$ *where* $v_n(y)$ *is defined by*

$$Vf(x, y) = (v_n(y)) = \left(\sum_{i=0}^{u} R_i^n \frac{\partial^i f(x, y)}{\partial x^i} \Bigg|_{x=0} \right).$$

It is easy to prove that

$$V(C_1 f_1(x, y) + C_2 f_2(x, y)) = C_1\, Vf_1(x, y) + C_2\, Vf_2(x, y)$$

where C_1 and C_2 are arbitrary constants.

THEOREM 1. If $Vf(x, y) = (v_n(y))$, *then we have the following relations*

$$1^0 \quad V \frac{\partial^m f(x, y)}{\partial x^m} = (D^m v_n(y))$$

$$2^0 \quad V y^k \frac{\partial^{m+p} f(x, y)}{\partial x^m \partial y^p} = \left(y^k D^m \frac{\partial^p v_n(y)}{\partial y^p} \right) \quad (m, p \in N_0)$$

$$3^0 \quad V \int_0^x f(t, y)\, dt = \left(\sum_{i=0}^{n-1} \prod_{j=i}^{n-1} (a + (j + 1) b)\, v_i(y) \right)$$

PROOF.

1^0 By definition 1 we have

$$V\,\frac{\partial^m f(x,y)}{\partial x^m}=\left(\sum_{i=0}^{n} R_i^n\,\frac{\partial^{i+m} f(x,y)}{\partial x^{i+m}}\bigg|_{x=0}\right)=$$

$$\left(\sum_{i=m}^{n+m} R_{i-m}^n\,\frac{\partial^i f(x,y)}{\partial x^i}\bigg|_{x=0}\right).$$

Since $D^m R_i^n = R_{i-m}^n$, we have that

$$D^m\,Vf(x,y)=\left(\sum_{i=m}^{n+m} R_{i-m}^n\,\frac{\partial^i f(x,y)}{\partial x^i}\bigg|_{x=0}\right) \text{ i, e.}$$

$$V\,\frac{\partial^m f(x,y)}{\partial x^m}=D^m Vf(x,y). \text{ which proves } 1^0.$$

2^0 The proof of 2^0 is similar to 1^0.

3^0 Suppose tha $V\int_0^x f\partial(t,y)\,dt=(w_n(y))$. By definition of V-transform we have

$$w_n(y)=\sum_{i=1}^{n} R_i^n\,\frac{\partial^{i-1} f(x,y)}{\partial x^{i-1}}\bigg|_{x=0}$$

and hence

(1) $$w_{n+1}(y)-(a+bn)\,w_n(y)=v_n(y),$$

Since the general solution of the equation (1) is the sequence

$\left(\sum_{i=0}^{n-1}\prod_{j=i}^{n-1}(a+(j+1)b)\,v_i(y)+C\right)$, where C is an arbitrary constant,

and $w_1(y)=v_0(y)$, we have 3^0.

By S_n we denote the set of sequences $\{(e_n(y))\}$ for which $D^k e_n(y)\,|_{n=0}$ $(k\in N_0)$ exists.

DEFINITION 2. E-transform of a sequence $(e_n(y)) \in S_n$ *is the function* $f(x, y)$ *defined by*

$$E(e_n(y)) = f(x, y) = \sum_{i=0}^{+\infty} \frac{D^k e_n(y)|_{n=0}}{k!} x^k.$$

THEOREM 2. Transforms V and E are inverše to each other.

The proof of theorem 2 is analogous to the proof of the correspo nding theorem from [5].

DEFINITION 3. R-transform of a function $f(x, y) \in C^\infty(R\,xR)$ *is the sequence* $Rf(x,y) = (r_{m,n})$ *where* $r_{m,n}$ *is defined by*

$$Rf(x, y) = (r_{m,n}) = \left(\sum_{i=0}^{m} \sum_{j=0}^{n} R_i^m R_j^n \frac{\partial^{i+j} f(x, y)}{\partial x^i \partial y^j} \Bigg|_{\substack{x=0\\ y=0}} \right).$$

THEOREM 3. If $Rf(x, y) = (r_{m,n})$, *then we have the following relations:*

$$1^0 \quad R \frac{\partial^{i+j} f(x, y)}{\partial x^i \partial y^j} = D_m^i D_n^j Nf(x,y)$$

$$2^0 \quad R \int_0^x f(t, y)dt = \left(\sum_{i=0}^{n-1} \prod_{j=i}^{n-1} (a + (j + 1)\, b) r_{i,n} \right).$$

By $S_{m,n}$ we denote the set of sequences $\{(i_{m,n})\}$ for which

$$D_m^i D_n^j i_{m,n} \big|_{m=0,\ n=0} \quad (i, j \in N_0) \text{ exist.}$$

DEFINITION 4. I-tranform of a sequence $(i_{m,n}) \in S_{m,n}$ *is the function* $f(x,y)$ *detined by*

$$I(i_{m,n}) = f(x, y) = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{D_m^i D_n^j \ i_{m,n} \big|_{m=0,\ n=0}}{i\,!\,j!} x^i y^j$$

assuming that the above sum exist.

THEOREM 4. Transform R and I are inverse to each other.

The proofs of theorem 3 and theorem 4 are similar to the proofs of the theorem 1 and the theorem 2 and we shall omit them.

3. SOME APPLICATIONS OF THE V AND R-TRANSFORM TO SOLVING DIFFERENCE EQUATIONS

In this part we will give some appli ations of the V, R and its inverse transformations in solving some differential-difference and partial difference equations.

EXAMPLE 1. By an application of the *E*-transform to the equation

(2) $$v'_{n+1}(y) - cv_{n+1}(y) - (a + d + bn)\, v'_n(y) + c\,(a + d + bn)\, v_n(y) = 0.$$

we get the equation

(3) $$f_{xy} - cf_x - df_y + cdf = 0.$$

Since the general solution of the equation (3) is given by

$$f(x, y) = f_1(y)\, e^{dx} + f_2(x)\, e^{cy},$$

where $f_1(y)$ and $f_2(x)$ are arbitrary functions, we have by an application of the *V*-transform that the general solution of (2) is given by

$$v_n(y) = f_1(y) \prod_{i=0}^{n-1} (d + a + bi) + f_n e^{cy}$$

where $f_1(y)$ is an arbitrary function and f_n is an arhitrary sequence

EXAMPLE 2. By an application of the *E*-tranform to the equati on

(4) $$yv'_{n+1}(y) - kv_{n+1}(y) - (1 + a + bn)\, yv'_n(y) + k\,(1 + a + bn)\, v_n(y) = 0$$ we

get the equation

(5) $$yf_{xy} - kf_x - yf_y + kf = 0.$$

Since the solution of the equation (5) is given by

$$f(x, y) = f_1(x)\, y^k + f_2(y)\, e^x,$$

where $f_1(x)$ an1 $f_2(y)$ are arbitrary functions, we have by an application of the *V*-transform that the general solution of (4) is given by

$$v_n(y) = f_n\, y^k + f(y) \prod_{i=0}^{n-1} (1 + a + ib),$$

where $f(y)$ is an arbitrary function and f_n is an arbritrary sequence.

EXAMPLE 3. By an application of the I-transform to the equation

(6) $$v_{m+1,\, n+1} - v_{m+2,\, n} + (1 + a + b\,(1 - n + 2m))\, v_{m+1,\, n} -$$

$$(1 + a + bm)\, v_{m,n+1} + b\,(1 + a + bm)\,(n\text{-}m)\, v_{m,\, n} = 0$$

we get the equation

(7) $$f_{xy} - f_{xx} + f_x - f_y = 0.$$

Since a particular solution of the eqration (7) is given by

$$f(x, y) = C\,(x + y)^k + f(y)\, e^x,$$

where $f(y)$ is an arbitrary function and C is an arbitrary constant, we have by an appli ation of the R-transform that a particular solution of (6) is given by

$$v_{m,n} = Ck! \sum_{i=0}^{k} R_i^m R_{k-i}^n + f_n \prod_{i=0}^{m-1} (1 + a + bi),$$

where f_n is an arbitrary sequence and C an arbritrary constant.

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Резиме

Во овој труд се разгледува линеарна трансформација која го пресликува множеството $C^\infty(R)$ функции $f: R^2 \to R$ кои имаат непрекинати парцијални изводи во однос на една променлива во множеството на реални низи $\{(v_n(y)\}$ со својство дека парцијалните изводи се

трансформираат во диференции D. Понатаму разгледуваме линеарна трансформација која го пресликува множеството C^{∞} $(R \times R)$ функции $f: R^2 \to R$ кои имаат непрекинати парцијални изводи од секој ред во однос и на двата аргумента во множеството реални низи $\{(V_{m,n})\}$ со својството дека парцијалните изводи се пресликуваат во парцијални диференции.

Овие трансформации, заедно со нивните инверзни трансформации, ги применуваме при решавање на некои диференцијално-диференцни равенки и парцијални диференцни равенки со воспоставување на аналогии меѓу овие равенки и соодветни диференцијални равенки.