Математички Билтен
23 (XLIX)
1999 (25–32)
Скопје, Македонија

ISSN 0351–336X

ON SOME CHARACTERISTIC SUBGROUPS OF THE GROUP OF UPPER TRIANGULAR MATRICES

S. Ќera and K. Trenčevski

Abstract

We define and describe a regular subgroup of the Lie group $K_n(F)$ of $n \times n$ upper triangular real or complex matrices with one on the main diagonal. We find that the number of such subgroups is $n!$ and we propose a construction of a graph over the set of these subgroups.

1. Introduction

Let F be the field of the real or complex numbers. By $K_n(F)$ we denote the Lie group of the $n \times n$ upper triangular real or complex matrices whose diagonal elements are one. For a given set S of some pairs of indices (i, j), $1 \leq i < j \leq n$, the corresponding entries of the matrices in $K_n(F)$ are called *fixed elements*, and all the other upper triangular entries are called *free elements*. The complement of S in the set of all those pairs (i, j) is denoted by S', i.e. $S' = \{(i,j) | 1 \leq i < j \leq n, \ (i,j) \notin S\}$. The set S induces a subset G of matrices in $K_n(F)$ whose all the fixed elements are zero, while the free elements are arbitrary elements of F. The subset G' of $K_n(F)$ defined by S' is called dual to G. We note that G' can be obtained from G by replacing the fixed elements by the free elements and vice versa. If S is such a set that both of the induced subsets G and G' are (Lie) subgroups of $K_n(F)$, then we call G to be a *regular subgroup* of $K_n(F)$. The regular subgroups are called *cells* also. Note that G is a regular subgroup of $K_n(F)$ if and only if G' is a regular subgroup of $K_n(F)$.

2. Main results

Let $M_n = \{1, \cdots, n\}$ and let ρ be a relation in M_n such that $j\rho i$ implies $j > i$. The dual relation ρ' in M_n is defined by $j\rho' i$ if and only if $j > i$ and $(j, i) \notin \rho$. For given ρ we join the set $S = \{(j, i) : i\rho j\}$. S induces a subset G of $K_n(F)$ and its dual G'.

Proposition 2.1. *The relation ρ corresponding to G is transitive if and only if its dual set G' is a subgroup of $K_n(F)$.*

Proof. Note that $(j, i) \notin \rho$ if and only if $j\rho' i$ which means that $a_{ij} \equiv 0$.

Now let us assume that ρ is transitive. Let A and B be two matrices of the set corresponding to ρ' and $AB = C(= [c_{pq}])$. Let $(j, i) \notin \rho$. For the element $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ if there exists k such that $a_{ik} b_{kj} \neq 0$, then $a_{ik} \neq 0$ and $b_{kj} \neq 0$ and hence $j\rho k$ and $k\rho i$. Since ρ is transitive, then $j\rho i$ - a contradiction with the assumption that $(j, i) \notin \rho$. Thus, $c_{ij} = 0$ which means that if ρ is transitive, then the corresponding dual set G' is a subgroupoid of $K_n(F)$.

Let A be a matrix from the set induced by ρ' and let B be the inverse matrix of A. Let $a_{ij} \equiv 0$, i.e. $j\rho' i$. Then we have to show that $b_{ij} \equiv 0$. Thus it will follow that if ρ is transitive, then the corresponding dual set G' will be a subgroup of $K_n(F)$.

As $j > i$, the Kronecker delta $\delta_{ij} = 0$, i.e. $\sum_{k=1}^{n} a_{ik} b_{kj} = 0$. If there exists k such that $a_{ik} b_{kj} \neq 0$, then $a_{ik} \neq 0$ and $b_{kj} \neq 0$. Hence $j\rho k$ and $k\rho i$, and since ρ is transitive, it follows that $j\rho i$ which contradicts to $j\rho' i$. Thus, for any $k \in \{1, \cdots, n\}$, $a_{ik} b_{kj} = 0$. Specially, for $k = i$, since $a_{ii} = 1$ b_{ij} must be zero.

Conversely, let us assume that the dual set G' is a subgroup of $K_n(F)$. Let $j\rho i$ and $i\rho k$, i.e. let a_{ij} and a_{ki} be non-zero. We choose a matrix A from the set corresponding to ρ', such that $a_{ks} = 0$ and $a_{sj} = 0$ for any s such that $k < s < j$ and $s \neq i$. Moreover, we assume that $a_{ij} \neq 0$ and $a_{ki} \neq 0$. The matrix $C = A^2 = AA$ belongs to the same set as A, because this set is a group by the assumption. So, we have

$$c_{kj} = \sum_{s=1}^{n} a_{ks} a_{sj} = a_{ki} a_{ij} \neq 0.$$

Since c_{kj} is non-zero, then $j\rho k$. Thus ρ is a transitive relation. ||

As a direct consequence we obtain the following proposition.

Proposition 2.2. *The cell G of $K_n(F)$ induced by a relation ρ is a subgroup of $K_n(F)$ if and only if the dual relation ρ' is transitive.*

Now we can prove the following theorem.

Theorem 2.3. *For any n, the number of cells in $K_n(F)$ is equal to $n!$ and the number of all relations ρ such that ρ and ρ' are transitive is equal to $n!$, too.*

Proof. From to the propositions 2.1 and 2.2 it follows that for any n the number of the cells in $K_n(F)$ is equal to the number of the relations ρ such that ρ and ρ' are transitive. So, it is sufficient to show that there exists a bijection between the set of relations ρ such that ρ and ρ' are transitive and the set S_n of all permutations τ on $M_n = \{1, \cdots, n\}$.

Let $\tau \in S_n$. We define a relation $\rho \in M_n$ by $i\rho j$ if $i > j$ and $\tau(i) < \tau(j)$. Then ρ' is defined by $i\rho' j$ if $i > j$ and $\tau(i) > \tau(j)$. It can be verified easily that ρ and ρ' are transitive. Conversely, let ρ be a relation in M_n such that ρ and ρ', defined as above, are transitive. By induction of n we can show that there exists (unique) permutation τ such that $i\rho j$ if and only if $i > j$ and $\tau(i) < \tau(j)$.

Let ρ be a relation on M_{n+1} such that ρ and ρ' are transitive. Then the restrictions $\bar{\rho}$ and $\bar{\rho}'$ of ρ and ρ' on the set M_n are also transitive and mutually dual. So, there exists (unique) permutation τ_n which induces the relations $\bar{\rho}$ and $\bar{\rho}'$. For any $i \in M_n$ only one of the possibilities $(n+1)\rho i$ and $(n+1)\rho' i$ is true, and since ρ and ρ' are transitive, then the permutation τ_n can be prolonged (uniquely) to a permutation $\tau_{n+1} : M_{n+1} \rightarrow M_{n+1}$ which has the required properties. So, the considered mapping $\tau \mapsto \rho$ is a bijection. ||

Note that if $\tau \mapsto \rho$, defined as in the proof of the theorem 2.3, then for the dual permutation τ', defined by $\tau'(i) = n + 1 - i$, $\tau' \mapsto \rho'$ is true.

The set of $n!$ relations ρ such that ρ and ρ' are transitive, can be parameterized as follows. Let the number of elements of the set

$$\{x : x \in M_n, \; j\rho x\}$$

be i_j for $2 \leq j \leq n$. Since $0 \leq i_j \leq j - 1$, then there are exactly $n!$ such sequences $(i_1, i_2, \cdots, i_n)$ $(i_1 = 0)$ and for any two different relations we have different sequences. Therefore, every such sequence corresponds to unique relation ρ. Also, it holds for the cells. So, we have proven the following theorem.

Theorem 2.4. *For any sequence $(i_1, i_2, \cdots, i_n)$, $\quad 0 \leq i_j \leq j - 1$ for $1 \leq j \leq n$, there exists unique relation ρ on M_n such that ρ and ρ' are transitive, and j is in relation ρ with i_j elements of M_n (i.e. there exists unique cell such that in the j-th column of its matrices there are exactly i_j elements equal to zero $(1 \leq j \leq n)$).* ||

The cell, whose matrices in the j-th column have exactly i_j free elements $(1 \leq j \leq n)$, is denoted by $C_{i_1 i_2 \cdots i_n}$. As a consequence of the Theorem 2.4 we obtain the following corollary.

Corollary 2.5. *Let ρ and ρ' be transitive relations on M_n. Then for any $t \in \{0, 1, \cdots, n\}$ there exist t elements $i_1, \cdots, i_t \in M_n$ unique up to* *permutation, such that by prolonging the relation ρ by $(n+1)\rho i_1, \cdots, (n+1)\rho i_t$ again we obtain transitive relations ρ and ρ' on M_{n+1}. Also, any cell of $n \times n$ matrices can be prolonged to a cell of $(n+1) \times (n+1)$ matrices by adding a new column with given number of fixed zeros in unique way.*

By the matrix mapping $(i,j) \to (n+1-j, n+1-i)$ every cell maps into cell. Indeed, if ρ and ρ' are transitive relations on the set M_n, then their inverse relations ρ^{-1} and ρ'^{-1} defined by

$$j\rho^{-1}i \iff i\rho j \quad \text{and} \quad j\rho'^{-1}i \iff i\rho' j,$$

also are transitive. Hence we obtain the following corollary.

Corollary 2.6. *Any cell of $n \times n$ matrices can be prolonged to a cell of $(n+1) \times (n+1)$ matrices by adding a new row with given number of fixed zeros in unique way.*

Note that the dimension of a cell as a Lie group, i.e. the number of the free elements, is equal to the number of all pairs (i,j) such that $i\rho' j$.

3. Graph over the regular subgroups $G_n(F)$

In the set $G_n(F)$ of all cells, t.e. regular subgroups, we define a relation " $>$ " such that $C_1 > C_2$ if C_2 can be obtained from C_1 by replacing one free element by fixed zero. Note that $dimC_1 - dimC_2 = 1$ and this relation can be extended up to transitive relation. Then, we prove the following proposition.

Proposition 3.1. *For any cell C there exist exactly $n-1$ cells C' such that $C > C'$ or $C' > C$. More precisely*

(i) if $dimC = n(n-1)/2$, then there exist $n-1$ cells C' such that $C > C'$,

(ii) if $dimC = 0$, then there exist $n-1$ cells C' such that $C' > C$,

(iii) if $0 < dimC < n(n-1)/2$, then there exist cells C' and C'' such that $C' > C > C''$. The number of such cells C' and C'' together is $n-1$.

Proof. One can verify that $C > C'$ or $C' > C$ if and only if the corresponding permutations τ and τ' are such that $\tau'(1)\tau'(2)\cdots\tau'(n)$ is obtained from $\tau(1)\tau(2)\cdots\tau(n)$ by a transposition of two neighbor elements $\tau(i)$ and $\tau(i+1)$. So, we get the first part of the proposition. The statements in (i) and (ii) are trivial. The statement in (iii) is a consequence of the following argument. If $\tau(1)\tau(2)\cdots\tau(n)$ is a permutation on M_n different from $12\cdots n$ and $n(n-1)\cdots 1$, then there exists at least one number $p \in \{1, 2, \cdots, n-1\}$ such that $\tau(p) < \tau(p+1)$ and there exists at least one number $q \in \{1, 2, \cdots, n-1\}$ such that $\tau(q) > \tau(q+1)$. ||

Now we give some examples of regular subgroups and the corresponding relation " $>$ " for $n = 2, 3, 4$. The free elements are denoted by $*$.

Example 1. For $n = 2$, there are two regular subgroups

$$C_{01} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}, \qquad C_{00} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the relation $>$ is given by

$$\begin{array}{c} C_{01} \\ \downarrow \\ C_{00} \end{array}$$

Example 2. For $n = 3$, the regular subgroups are the following

$$C_{012} = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \qquad C_{000} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C_{002} = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \qquad C_{010} = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C_{011} = \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C_{001} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix};$$

the relation $>$ is given by

$$\begin{array}{ccccc} & & C_{012} & & \\ & \swarrow & & \searrow & \\ C_{002} & & & & C_{011} \\ \downarrow & & & & \downarrow \\ C_{001} & & & & C_{010} \\ & \searrow & & \swarrow & \\ & & C_{000} & & \end{array}$$

Example 3. For $n = 4$, the regular subgroups are the following

$$C_{0123} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0000} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0023} = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0100} = \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0113} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0010} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0122} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0001} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0013} = \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0110} = \begin{bmatrix} 1 & * & * & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0022} = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0101} = \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0103} = \begin{bmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0020} = \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0112} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0001} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

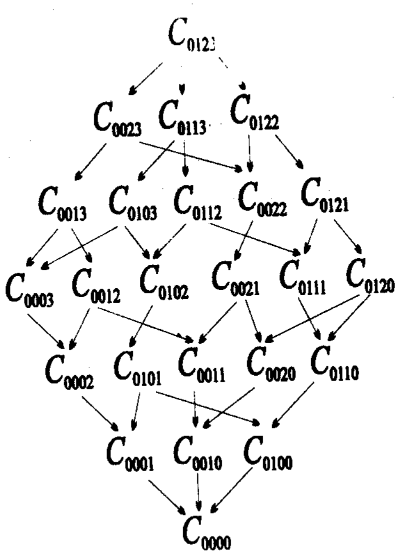
$$C_{0121} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0002} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0012} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{0111} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0003} = \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_{0120} = \begin{bmatrix} 1 & * & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_{0021} = \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_{0102} = \begin{bmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

the relation $>$ is given by



References

[1] A. A. Borisenko and Yu. A. Nikolaevskii, *Grassmann manifolds and Grassmann image of submanifolds*, Uspehi Mat. Nauk, **46** (1991), 41-83 (in Russian).

[2] K. Trenčevski, *Generalization of the Grassmann Manifolds, accepted for publication in the Proceeding of the Second conference of Balkan geometers*, Thessaloniki 24-27 June, 1998. submitted for publication in Balkan Journal of Geometry and Its Applications.

ЗА НЕКОИ КАРАКТЕРИСТИЧНИ ПОДГРУПИ НА ГРУПАТА ОД ГОРНОТРИАГОЛНИ МАТРИЦИ

С. Кера и К. Тренчевски

Резиме

Во трудот е дефинирана класа од подгрупи на Лиевата група од горнотриаголни $n \times n$ матрици со единици по дијагоналата, наречени регуларни подгрупи. Се докажува дека бројот на таквите подгрупи е $n!$ и се воведува ралација $>$ во множеството на овие подгрупи.

Institute of Mathematics
P.O.Box 162, 1000 Skopje
Macedonia