Math. Maced.
Vol. 7 (2009)
1–7

NOTE ON AN INTEGRAL INEQUALITY SIMILAR TO QI'S INEQUALITY

A. AGLIĆ ALJINOVIĆ* AND J. PEČARIĆ**

Abstract. Improvement and generalization via integral power means of an recently published integral inequality similar to Qi's inequality are given.

1. Introduction

The following problem was formulated by F.Qi in 2000. in his paper [3]: under what conditions does the inequality

$$\int_a^b [f(x)]^t\,dx \geq \left(\int_a^b f(x)\,dx\right)^{t-1} \tag{1.1}$$

hold for $t > 1$.

Since then this problem has been intensively considered and applied in Probability and Statistics (see for instance [4], [5], [6], [7], [8], [9]).

Also, in the recent paper [1] the following problem similar to the (1.1) was posed: under what conditions does the inequality

$$\int_a^b [f(x)]^t\,dx \leq \left(\int_a^b f(x)\,dx\right)^{1-t} \tag{1.2}$$

hold for $t < 1$.

In the same paper the following answers were given:

Theorem 1. *Let f and g be nonnegative functions with $0 < m \leq f(x)/g(x) \leq M < \infty$ for $x \in [a,b]$. Then for $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}}\,dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}}\,dx \tag{1.3}$$

and then

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}}\,dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\,dx\right)^{\frac{1}{p}}. \tag{1.4}$$

2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Qi's inequality, Hölder inequality.

Corollary 1. *For a given positive integer* $p > 2$ *and* f *a such that* $0 < m \leq f(x) \leq M < \infty$ *for* $x \in [a,b]$ *where* $M \leq m^{(p-1)^2}/(b-a)^p$, *we have*

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \leq \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

Remark 1. *In the last Remark an obvious typewriting mistake has been made: instead of "positive integer* $p > 2$*" should be " positive real number* $p > 1$*".*

The improvements of the integral inequalities from the Theorem 1 and Corollary 1 are given in the Section 2. Also, further generalization of the inequality (1.2) similar to Qi's are obtained in the Section 3. by using the integral power means.

2. Improvements of the inequality (1.4)

In the following two Theorems we give the improvements of the inequality (1.4).

Theorem 2. *Let* f *and* g *be positive functions with* $0 < f(x)/g(x) \leq M < \infty$ *for* $x \in [a,b]$. *Then for* $p, q \in \mathbb{R}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} - \frac{1}{q} > 0$ *and* $p, q > 0$ *we have*

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}}\,dx \leq M^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\,dx\right)^{\frac{1}{p}} \tag{2.1}$$

If $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} - \frac{1}{q} > 0$ *and* $p \cdot q < 0$ *inequality is reversed.*

Proof. We have

$$f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}} = f(x)^{\frac{1}{q}} g(x)^{\frac{1}{p}} \left(\frac{f(x)}{g(x)}\right)^{\frac{1}{p}-\frac{1}{q}}.$$

Since $\frac{1}{p} - \frac{1}{q} > 0$ and $0 < f(x)/g(x) \leq M$, we have

$$\left(\frac{f(x)}{g(x)}\right)^{\frac{1}{p}-\frac{1}{q}} \leq M^{\frac{1}{p}-\frac{1}{q}}$$

and

$$f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}} \leq M^{\frac{1}{p}-\frac{1}{q}} f(x)^{\frac{1}{q}} g(x)^{\frac{1}{p}}.$$

Thus

$$\int_a^b f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}}\,dx \leq M^{\frac{1}{p}-\frac{1}{q}} \int_a^b f(x)^{\frac{1}{q}} g(x)^{\frac{1}{p}}\,dx.$$

Further, if $\frac{1}{p} + \frac{1}{q} = 1$ by applying the Hölder inequality, in case $p, q > 0$ we have

$$\int_a^b f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}}\,dx \leq M^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\,dx\right)^{\frac{1}{p}}.$$

If $p \cdot q < 0$ inequality is reversed. □

Theorem 3. *Let f and g be positive functions with $0 < m \le f(x)/g(x) < \infty$ for $x \in [a,b]$. Then for $p,\ q \in \mathbb{R}$ and $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}-\frac{1}{q}<0$ and $p,\ q>0$ we have*

$$\int_a^b f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}}\,dx \le m^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\,dx\right)^{\frac{1}{p}} \tag{2.2}$$

If $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}-\frac{1}{q}<0$ and $p\cdot q<0$ inequality is reversed.

Proof. As in the previous Theorem

$$f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}} = f(x)^{\frac{1}{q}} g(x)^{\frac{1}{p}} \left(\frac{f(x)}{g(x)}\right)^{\frac{1}{p}-\frac{1}{q}}.$$

Since $\frac{1}{p}-\frac{1}{q}<0$ and $0<m\le f(x)/g(x)$, we have

$$\left(\frac{f(x)}{g(x)}\right)^{\frac{1}{p}-\frac{1}{q}} \le m^{\frac{1}{p}-\frac{1}{q}}$$

and

$$f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}} \le m^{\frac{1}{p}-\frac{1}{q}} f(x)^{\frac{1}{q}} g(x)^{\frac{1}{p}}.$$

Thus

$$m^{\frac{1}{p}-\frac{1}{q}} \int_a^b f(x)^{\frac{1}{q}} g(x)^{\frac{1}{p}}\,dx \ge \int_a^b f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}}\,dx.$$

Further, if $\frac{1}{p}+\frac{1}{q}=1$ by applying the Hölder inequality, in case $p,\ q>0$ we have

$$\int_a^b f(x)^{\frac{1}{p}} g(x)^{\frac{1}{q}}\,dx \le m^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\,dx\right)^{\frac{1}{p}}.$$

If $p\cdot q<0$ inequality is reversed. □

Remark 2. *Comparing to Theorem 1, Theorems 2 and 3 hold for a wider class of functions. However, (under the assumptions of Theorem 1) the constant in (2.1) is better than the constant in (1.4). Indeed, since $\frac{1}{M}\le\frac{1}{m}$ and $\frac{1}{p}+\frac{1}{q}=1$, $p,q>0$ we have*

$$M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} = M^{\frac{1}{p^2}} \left(\frac{1}{m}\right)^{\frac{1}{q^2}} \ge M^{\frac{1}{p^2}} \left(\frac{1}{M}\right)^{\frac{1}{q^2}} = M^{\frac{1}{p^2}-\frac{1}{q^2}} = M^{\frac{1}{p}-\frac{1}{q}}.$$

Also, the constant in (2.2) is better than the constant in (1.4) since

$$M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \ge m^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} = m^{\frac{1}{p^2}-\frac{1}{q^2}} = m^{\frac{1}{p}-\frac{1}{q}}.$$

Consequently, the next two Corollaries are the improvements of the Corollary1.

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<f(x)\le M<\infty$ for $x\in[a,b]$. Let also $p\in\mathbb{R}$, $1<p<2$ and $M\le(b-a)^{\frac{1}{p-2}}$, then we have*

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \le \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

Proof. We use the results from the Theorem 2 with $g(x) \equiv 1$. If $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}-\frac{1}{q}>0$ and $p,\, q>0$, then $q>2>p>1$. Putting $g(x)\equiv 1$ into (2.1) yields to

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \le M^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}}.$$

Since $\frac{1}{p-2}=\frac{q}{p-q}$ and $M\le (b-a)^{\frac{q}{p-q}}$ is equivalent to $M^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\le 1$ we have

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \le \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

□

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<m\le f(x)<\infty$ for $x\in[a,b]$. Let also $p\in\mathbb{R}$, $p>2$ and $m\ge (b-a)^{\frac{1}{p-2}}$, then we have*

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \le \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

Proof. We use the results from the Theorem 3 with $g(x)\equiv 1$. If $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}-\frac{1}{q}<0$ and $p,\, q>0$, then $p>2>q>1$. Putting $g(x)\equiv 1$ into (2.2) yields to

$$\int_a^b f(x)^{\frac{1}{p}}\,dx \le m^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}}$$

Since $\frac{1}{p-2}=\frac{q}{p-q}$ and $m\ge (b-a)^{\frac{q}{p-q}}$ is equivalent to $m^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\le 1$ we have

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \ge \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

□

In the next two Corollaries the reversed inequality of (1.2) is considered.

Corollary 4. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<f(x)\le M<\infty$ for $x\in[a,b]$. Let also $p\in\mathbb{R}$, $0<p<1$ and $M\ge (b-a)^{\frac{1}{p-2}}$, then we have*

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \ge \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

Proof. We use the result from the Theorem 2 in case $p\cdot q<0$ with $g(x)\equiv 1$. If $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}-\frac{1}{q}>0$ and $p\cdot q<0$, then $0<p<1$. Putting $g(x)\equiv 1$ into reversed (2.1) yields to

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \ge M^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}}$$

Since $\frac{1}{p-2}=\frac{q}{p-q}$ and $M\geq (b-a)^{\frac{q}{p-q}}$ is equivalent to $M^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\geq 1$ we have

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \geq \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

□

Corollary 5. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<m\leq f(x)<\infty$ for $x\in[a,b]$. Let also $p\in\mathbb{R}$, $p<0$ and $m\leq (b-a)^{\frac{1}{p-2}}$, then we have*

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \geq \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

Proof. We use the result from the Theorem 3 in case $p\cdot q<0$ with $g(x)\equiv 1$. If $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}-\frac{1}{q}<0$ and $p\cdot q<0$, then $p<0$. Putting $g(x)\equiv 1$ into reversed (2.2) yields to

$$\int_a^b f(x)^{\frac{1}{p}}\,dx \geq m^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\left(\int_a^b f(x)\,dx\right)^{\frac{1}{q}}$$

Since $\frac{1}{p-2}=\frac{q}{p-q}$ and $m\leq (b-a)^{\frac{q}{p-q}}$ is equivalent to $m^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}\geq 1$ we have

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \geq \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}.$$

□

3. Generalizations via power means

For a function $f:[a,b]\to\mathbb{R}$ integral power mean is defined by

$$M^{[r]}(f)=\left(\frac{1}{b-a}\int_a^b f^r(x)\,dx\right)^{\frac{1}{r}}.$$

In the following Theorem we give the further generalization of the inequality (1.2) similar to Qi's by using the integral power means.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<f(x)\leq M<\infty$ for $x\in[a,b]$. If $0<r<s,t$ and $M\leq (b-a)^{\frac{r-s}{s(t-r)}}$ we have*

$$\int_a^b f^t(x)\,dx \leq \left(\int_a^b f^s(x)\,dx\right)^{\frac{r}{t}}. \tag{3.1}$$

Proof. Since $f(x)\leq M$ and $t-r>0$ we have

$$f^t(x)=f^{t-r}(x)\cdot f^r(x)\leq M^{t-r}f^r(x),$$

and for $s>r>0$ we have $M^{[r]}(f)\leq M^{[s]}(f)$ (see for instance [2]). Thus

$$\frac{1}{b-a}\int_a^b f^t(x)\,dx \leq M^{t-r}\left[\left(\frac{1}{b-a}\int_a^b f^r(x)\,dx\right)^{\frac{1}{r}}\right]^r$$
$$\leq M^{t-r}\left(\frac{1}{b-a}\int_a^b f^s(x)\,dx\right)^{\frac{r}{s}}$$

and

$$\int_a^b f^t(x)\,dx \leq M^{t-r}(b-a)^{1-\frac{r}{s}}\left(\int_a^b f^s(x)\,dx\right)^{\frac{r}{s}}.$$

Since $M\leq(b-a)^{\frac{r-s}{s(t-r)}}$ is equivalent to $M^{t-r}(b-a)^{1-\frac{r}{s}}\leq 1$ the inequality (3.1) is proved. □

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<m\leq f(x)<\infty$ for $x\in[a,b]$. If $0<t<r<s$ and $m\geq(b-a)^{\frac{r-s}{s(t-r)}}$ we have*

$$\int_a^b f^t(x)\,dx \leq \left(\int_a^b f^s(x)\,dx\right)^{\frac{r}{t}}. \tag{3.2}$$

Proof. Since $f(x)\geq m$ and $t-r<0$ we have

$$f^t(x)=f^{t-r}(x)\cdot f^r(x)\leq m^{t-r}f^r(x),$$

and for $s>r>0$ we have $M^{[r]}(f)\leq M^{[s]}(f)$. Thus

$$\frac{1}{b-a}\int_a^b f^t(x)\,dx \leq m^{t-r}\left[\left(\frac{1}{b-a}\int_a^b f^r(x)\,dx\right)^{\frac{1}{r}}\right]^r$$
$$\leq m^{t-r}\left(\frac{1}{b-a}\int_a^b f^s(x)\,dx\right)^{\frac{r}{s}}$$

and

$$\int_a^b f^t(x)\,dx \leq m^{t-r}(b-a)^{1-\frac{r}{s}}\left(\int_a^b f^s(x)\,dx\right)^{\frac{r}{s}}.$$

Since $m\geq(b-a)^{\frac{r-s}{s(t-r)}}$ is equivalent to $m^{t-r}(b-a)^{1-\frac{r}{s}}\leq 1$ the inequality (3.2) is proved. □

Corollary 6. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $0<f(x)\leq M<\infty$ for $x\in[a,b]$. If $t\in\left\langle\frac{1}{2},1\right\rangle$ and $M\leq(b-a)^{\frac{t}{1-2t}}$ we have*

$$\int_a^b f^t(x)\,dx \leq \left(\int_a^b f(x)\,dx\right)^{1-t}. \tag{3.3}$$

Proof. We take $t \in \left\langle \frac{1}{2}, 1 \right\rangle$, $r = 1 - t$, $s = 1$ in the Theorem 4. Thus we have $0 < r < s, t$ and $\frac{r-s}{s(t-r)} = \frac{t}{1-2t}$, so inequality (3.1) reduces to (3.3). □

Corollary 7. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $0 < m \leq f(x) < \infty$ for $x \in [a, b]$. If $t \in \left\langle 0, \frac{1}{2} \right\rangle$ and $m \geq (b-a)^{\frac{t}{1-2t}}$ we have*

$$\int_a^b f^t(x)\, dx \leq \left(\int_a^b f(x)\, dx \right)^{1-t}. \tag{3.4}$$

Proof. We take $t \in \left\langle 0, \frac{1}{2} \right\rangle$, $r = 1 - t$, $s = 1$ in the Theorem 5. Thus we have $0 < t < r < s$ and $\frac{r-s}{s(t-r)} = \frac{t}{1-2t}$, so inequality (3.2) reduces to (3.4). □

Remark 3. *It is easy to see that if we take $t = \frac{1}{p}$ in the Corollary 6 and Corollary 7 we obtain Corollary 2 and Corollary 3 respectively.*

Acknowledgement 1. *This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 036-1170889-1054 (first author) and 117-1170889-0888 (second author).*

References

[1] L. Bougoffa, *An integral inequality similar to Qi's inequality,* J. Inequal. Pure Appl. Math., Vol. 6, Issue 1, Article 27, (2005).
[2] D. S. Mitrinović, J. E. Pecarić and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.
[3] F. Qi, *Several integral inequalies,* J. Inequal. Pure Appl. Math., Vol. 1, Issue 2, Article 19, (2000).
[4] L. Bougoffa, *Notes on Qi's type inequalities,* J. Inequal. Pure Appl. Math., Vol. 4, Issue 4, Article 77, (2003).
[5] V. Csiszar and T. F. Móri, *The convexity method of proving moment-type inequalities,* Statist. Probab. Lett., Vol. 66, (2004), 303-313.
[6] S. Mazouzi and F. Qi, *On a open problem regarding an integral inequality,* J. Inequal. Pure Appl. Math., Vol. 4, Issue 2, Article 31, (2003).
[7] J. Pecarić and T. Pejković, Note *on* F.*Qi's integral inequality,* J. Inequal. Pure Appl. Math., Vol. 5, Issue 3, Article 51, (2004).
[8] T. K. Pogány, *On an open problem of* F.*Qi,* J. Inequal. Pure Appl. Math., Vol. 3, Issue 4, Article 54, (2002).
[9] A. Witkowski, *On a* F.*Qi's integral inequality,* J. Inequal. Pure Appl. Math., Vol. 6, Issue 2, Article 36, (2005).

*Department of applied mathematics,
Faculty of Electrical Engineering and Computing,
University of Zagreb,
Unska 3, 10 000 Zagreb, Croatia
E-mail address: andrea.aglic@fer.hr

**Faculty of textile technology,
University of Zagreb,
Pierottijeva 6, 10000 Zagreb, Croatia
E-mail address: pecaric@hazu.hr