Математички Билтен
28 (LIV)
2004 (133–138)
Скопје, Македонија

ISSN 0351–336X

CONTINUOUS FLOW ON UNIFORM STRUCTURE OF FAMILY $F(T,X)$

Neki Dervishi

Abstract

Let $F(T,X)$ be a family of all continuous functions defined on T with values in X, where T is Hausdorff topological abelian group, locally compact, $(Y,\mathcal{V})$ Hausdorff uniform space and $X = Y^T$. The family $F(T,X)$ we can endow with the relativ uniformity of uniform convergence on compacta. The mapping $\phi: F \times T \to F$ defined by $\phi(f,t) = f_t$, where $f_t(s) = f(t \otimes s)$, defines a continuous flow on structure $F(T,X)$.

1. Introduction

Let T be a Hausdorff topological Abelian group, locally compact and $(Y,\mathcal{V})$ Hausdorff uniform space. If we denote

$$W(M) = \{(f,g) \in (Y^T)^2 / \big(f(t), g(t)\big) \in M, \quad \forall t \in T\}$$

$$\mathcal{B} = \{W(M)/M \in \mathcal{V}\} \subset \mathcal{U}$$

then the family $\mathcal{B}$ is a base for $\mathcal{U} = U_T \subset P\big((Y^T)^2\big)$-uniformity of uniform convergence (on $X = Y^T$) and the pair $(X = Y^T, \mathcal{U})$ is Hausdorff uniform space. If we denote

$$R(M) = \{(f,g) \in (X^T)^2 / \big(f_s(t), g_s(t)\big) \in M, \quad \forall s,t \in T\}$$

$$\mathcal{P} = \{R(M)/M \in \mathcal{V}\}$$

then the family $\mathcal{P}$ is a base for $\mathcal{M} \subset P\big((X^T)^2\big)$ - uniformity of uniform convergence (on X^T) and the pair $(X^T, \mathcal{M})$ is Hausdorff uniform space.

The subfamily $F(T,X) \subset X^T$ we can endow with the relativ uniformity of uniform convergence an compacta. Let $\ell \subset P(T) = 2^T$ be a family of all compact subset on T. If we denote

$$M_A(E) = \{(f,g) \in \big(F(T,X)\big)^2 / \big(f_s(t), g_s(t)\big) \in E, \quad \forall s \in T, \quad \forall t \in A\}$$

$$\mathbf{Z} = \{M_A(E)/A \in \ell, \quad E \in \mathcal{V}\} \subset \mathcal{R}/\ell$$

then the family $\mathbf{Z}$ is a subbase for $\mathcal{R}/\ell \subset P\big((F(T,X))^2\big)$, - relativ uniformity of uniform convergence on compacta and the pair $\big(F(T,X), \mathbf{Z}\big)$ is Hausdorff uniform space.

Definition. *Let $\mathcal{H}$ be a topological space and $\mathcal{G}$ any algebraic group. The mapping $\varphi: \mathcal{H} \times \mathcal{G} \to \mathcal{H}$ is said to be a General flow (or general dynamical system) on $\mathcal{H}$, if satisfying the following axioms:*
(a_1) (*Identity property*)

$$\varphi(x,0) = x, \quad \forall x \in \mathcal{H} \quad \text{(where 0 is the identity of } \mathcal{G}).$$

(a_2) (*Group property*)

$$\varphi\big(\varphi(x,t), s\big) = \varphi(x, t \oplus s) \quad \forall x \in \mathcal{H} \;\&\; \forall t, s \in \mathcal{G}.$$
$$\text{(where } \oplus \text{ is the group operation of } \mathcal{G})$$

(a_3) (*Continuity property*).
The mapping $\varphi: \mathcal{H} \times \mathcal{G} \to \mathcal{H}$ is continuous in $\mathcal{H}$. In other words, for each neighborhood S of point $\varphi(x,t)$ there exist a neighborhood E of $x \in \mathcal{H}$ such that $\phi(E,t) \subseteq S$.

The general flow $\varphi: \mathcal{H} \times \mathcal{G} \to \mathcal{H}$ on $\mathcal{H}$, is said to be *continuous flow* on $\mathcal{H}$, if $\mathcal{G}$ is a topological group and φ is continuous in $\mathcal{H} \times \mathcal{G}$.

2. The result

Let $F(T,X)$ be a family of all continuous functions $f: T \to X$. Suppose that $F(T,X)$ has $\mathcal{K}/\ell \subset P\big((F(T,X))^2\big)$ - the relativ uniformity of uniform convergence on compacta. If the mapping $\phi: F \times T \to F$ defined by

$$\forall (f,t) \in F \times T, \quad \phi(f,t) = f_t, \quad f_t \in F$$

where $f_t(\theta) = f(\theta \oplus t)$, $\forall t, \theta \in T$, then satisfied theorem:

Theorem. *Let T be a Hausdorff topological abelian group locally compact and Y Hausdorff uniform space. The mapping $\phi(f,t) = f_t$ defines a continuous flow on uniform structure $F(T,X)$.*

Proof. We shall prove the axioms of flow.

(a_1) (Identity property). By definition

$$\begin{aligned}&\phi\big(f(\theta),t\big) = f_t(\theta) = f(\theta \oplus t)\\&\phi\big(f(\theta),0\big) = f_\circ(\theta) = f(\theta), \quad \forall\,\theta \in T\\&\phi(f,0) = f, \quad \forall\, f \in F(T,X).\end{aligned}$$

(a_2) (Group property). For each $t, s \in T$ and for each $f \in F(T,X)$.

$$\begin{aligned}&\phi\big(f(\theta),t\big) = f_t(\theta) = f(\theta \oplus t)\\&\phi\big(\phi(f(\theta),t),s\big) = \phi\big(f(t\oplus\theta),s\big) = f_s(t\oplus\theta)\\&f_s(t\oplus\theta) = f(t\oplus s\oplus\theta) = f_{t\oplus s}(\theta)\\&\phi\big(\phi(f(\theta),t),s\big) = f_{t\oplus s}(\theta), \quad \forall\,\theta\in T\end{aligned}$$

or

$$\left.\begin{aligned}\phi\big(\phi(f,t),s\big) = f_{t\oplus s}\\ f_{t+s} = \phi(f,t\oplus s)\end{aligned}\right\} \Rightarrow \phi\big(\phi(f,t),s\big) = \phi(f,t\oplus s).$$

(a_3) (Continuity property). Let us now show that mapping $\phi\colon F\times T \to F$ is continuous function. Assume that family $F(T,X)$ has $\mathcal{K}/\ell \subset P\big((F(T,X))^2\big)$ - the relativ uniformity of uniform convergence on compacta. That is

$$\begin{aligned}&W_B(E) \in \mathcal{K}/\ell \Leftrightarrow W_B(E) =\\&= \{(f,g) \in (F(T,X))^2/(f_s(\theta),g_s(\theta)) \in E, \quad \forall\, s\in T, \quad \forall\,\theta\in B\}\\&W_B(E)(f) = \{g\in F(T,X)/(f_s(\theta),g_s(\theta))\in E, \quad \forall\, s\in T, \quad \forall\,\theta\in B\}.\end{aligned}$$

Let $(f,t) \in F\times T$ be an arbitrary point and $B \subset T$ compact subgroup. who is a neighborhood of a point $t \in B$. Let $\{s_a, a\in \mathbf{D}\} \subset B$ be a net in B, which converges to a point 0 (where 0-is the identity of B & T). In other words $\forall\,\theta, s_a \in B \Rightarrow \theta\oplus s_a \in B$, (where $\oplus$ is the group operation of B) and the net $\{t\oplus s_a,\ a\in\mathbf{D}\}\subset T$ converges to a unique point $t\in T$, because T is Hausdorff space. Let $\{f^{(a)},\ a\in\mathbf{D}\}\subset F(T,X)$ be a net in $F(T,X)$ which converges to a point $f\in F(T,X)$. The point $f\in F(T,X)$ is unique, because the family $F(T,X)$ is Hausdorff uniform space, relative to the topology of relativ uniformity of uniform convergence on compacta.

The net $\{(f^{(a)},t\oplus s_a), a\in\mathbf{D}\}\subset F\times T$ converges to a unique point $(f,t)\in F\times T$, because the family $F\times T$ is Hausdorff uniform space, relative to the product topology (cf. Kelley [1, p. 91, Theorem 4.]).

For continuity of function ϕ, it will suffice to show that the corresponding net $\{\phi((f^{(a)},t\oplus s_a),\ a\in\mathbf{D}\}\ \subset F(T,X)$ converges to a unique point

$\phi(f,t) \in F(T,X)$, (cf., R. A. Aleksandran & E. A. Miržahanan [4. p. 99, Theorem 4.5]). In other words, it will suffice to show that

$$\phi(f^{(a)}, t \oplus s_a) \to \phi(f,t) \tag{1}$$

relative to the topology of relativ uniformity of uniform convergence on compacta. By definition

$$\left.\begin{aligned} \phi(f^{(a)}, t \oplus s_a) = (f_{t\oplus s_a})^{(a)} \\ \phi(f,t) = f_t \end{aligned}\right\} \Rightarrow (f_{t\oplus s_a})^{(a)} \to f_t(u\mathcal{J}(\mathcal{K}/\ell)) \Leftrightarrow$$

$$\Leftrightarrow \quad \left.\begin{aligned} (\forall W_B(E) \in \mathcal{K}/\ell)(\exists m \in \mathbf{D})\ (\forall a \in \mathbf{D}) \\ a > m \Rightarrow (f_{t\oplus s_a})^{(a)} \in W_B(E)(f_t) \end{aligned}\right\} \tag{2}$$

By suppose $f^{(a)} \to f$, in topology $T\big(R/(C)\big)$ of family $F(T,X)$. That is

$$\left.\begin{aligned} (\forall W_B(E) \in \mathcal{K}/\ell)(\exists m \in \mathbf{D})(\forall a \in \mathbf{D} \\ a > m \Rightarrow f^{(a)} \in W_B(E)(f) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (f, f^{(a)}) \in W_B(E) \Leftrightarrow \big((f_s(\theta), (f_s)^{(a)}(\theta)\big) \in E, \quad \forall s \in T, \quad \forall \theta \in B).$$

By definition $B \subset T$ is compact subgroup of group T, therefore

$$\big(f_s(\theta \oplus s_a), (f_s)^{(a)}(\theta \oplus s_a)\big) \in E, \quad \forall s \in T, \quad \forall \theta \in B. \tag{3}$$

The net $\{\theta \oplus s_a, a \in \mathbf{D}\} \subset B$ converges to a point $\theta \in B$. Assume that $f \in F(T,X)$ is a continuous function, therefore the net $\{f(\theta \oplus s_a), a \in \mathbf{D}\} \subset X$ converges to a point $f(\theta) \in X$. In other words

$$\left.\begin{aligned} (\forall W(E) \in \mathcal{U} = U_T)(\exists m \in \mathbf{D})(\forall a \in \mathbf{D}) \\ a > m \Rightarrow f(\theta \oplus s_a) \in W(E)\big(f(\theta)\big) \end{aligned}\right\} \Leftrightarrow \tag{4}$$

$$\Leftrightarrow \big(f(\theta), f(\theta \oplus s_a)\big) \in W(E) \Leftrightarrow \big(f_s(\theta), f_s(\theta \oplus s_a)\big) \in E, \ \forall s \in T, \ \forall \theta \in B.$$

It follows from the relations (4) & (3) that

$$\left.\begin{aligned} \big(f_s(\theta), f_s(\theta \oplus s_a)\big) \in E \\ \big(f_s(\theta \oplus s_a), (f_s)^{(a)}(\theta \oplus s_a)\big) \in E \end{aligned}\right\} \Rightarrow \big(f_s(\theta), (f_s)^{(a)}(\theta \oplus s_a)\big) \in E \cdot E.$$

By suppose $(Y, \mathcal{V})$ is Hausdorff uniform space, therefore

$$(\forall Q \in \mathcal{V})(\exists E \in \mathcal{V}), \quad E \cdot E \subseteq Q.$$

In other words, this show that

$$\big((f_s(\theta),(f_s)^{(a)}(\theta\oplus s_a))\in Q \quad \forall s\in T,\quad \forall\theta\in B\big) \Rightarrow$$

$$\Rightarrow\ \big((f_s(\theta),(f_{s\oplus s_a})^{(a)}(\theta))\in Q,\quad \forall t\in T,\quad \forall\theta\in B\big).$$

Assume that T is the abelian group, therefore

$$\forall t,s\in T\ \Rightarrow\ t\oplus s\in T.$$

In other words, it follows that

$$\big((f_{t\oplus s}(\theta),(f_{t\oplus s\oplus s_a})^{(a)}(\theta))\in Q,\quad \forall s,t\in T,\quad \forall\theta\in B\big) \Leftrightarrow$$

$$\Leftrightarrow\ \big((f_t,(f_{t\oplus s_a})^{(a)}\big)\in W_B(Q)\ \Leftrightarrow\ (f_{t\oplus s_a})^{(a)}\in W_B(Q)(f_t).$$

That is

$$\left.\begin{array}{r}(\forall W_B(Q)\in\mathcal{K}/\ell)(\exists m\in\mathbf{D})(\forall a\in\mathbf{D})\\ a>m\ \Rightarrow\ (f_{t\oplus s_a})^{(a)}\in W_B(Q)(f_t)\end{array}\right\}\Leftrightarrow\ (f_{t\oplus s_a})^{(a)}\to f_t$$

$$\left.\begin{array}{r}(f_{t\oplus s_a})^{(a)}=\phi(f^{(a)},t\oplus s_a)\\ f_t=\phi(f,t)\end{array}\right\}\Rightarrow\ \phi(f^{(a)},t\oplus s_a)\to\phi(f,t).\qquad\square$$

References

[1] I. J. Kelley: *General Topology,* Springer-Verlag, New York, Heidelberg, Berlin, 1955.

[2] G. R. Sell: *Topological Dynamics and ordinary differential equations,* Van Nostrand Reinhold Company, London 1971.

[3] K. S. Sibirski: *Videnie Vtopologičeskuju dinamiku,* Kišinev, 1970.

[4] R. A. Aleksandran, & E. A. Miržahanan, Topologija, Moskva 1979.

НЕПРЕКИНАТ ТЕК ВО РАМНОМЕРНАТА СТРУКТУРА НА ФАМИЛИЈАТА $F(T, X)$

Neki Dervishi

Резиме

Користејќи ја дефиницијата за непрекинат тек во произволен тополошки простор (види [3]) докажана е следнава теорема: Ако T е локално компактна Хаусдорфова Абелова топлошка група и Y е рамномерен Хаусдорфов простор, тогаш функцијата $\phi(f, t) = f_t$ дефинира еден непрекинат тек во структурата $F(T, X)$.

Department of mathematics

University of Prishtina

38000 Prishtina

Kosovo