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SOME CHARACTERISTICS OF OUTPUT STREAM OF UNSERVED CUSTOMERS IN $G^X/G/1$ AND $G_D^X/G_D/1$ SYSTEM

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Abstract

In this paper, an output stream of unserved customers is considered. Namely, during the service of a customer or a group of customers, other customers which wait in the system lose some properties and cannot be accepted for servicing. Therefore, they leave the system unserved. We determinate the Laplace-Stieltjes transformation and the probability generating function of the inter-output times between two groups of unserved customers in $G^X/G/1$ and $G_D^X/G_D/1$ system, respectively. As a special case, we find the distribution of this characteristic for $M/M/1$ and $Geo/Geo/1$ systems.

1. Introduction.

In the paper [2] the output process of groups of unserved customers generated during the service time in the system $M/G/1$ are considered. Here, we generalize the problem and determine the Laplast-Stieltjes transform (LST) and the probability generating function (p.g.f) of the length of the interval between output moments of two consecutive unserved groups.

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In *Section 2* we describe the systems which are considered. In *Section 3* we give some preliminary results about the number of the customers arrived in the system during the service time of a customer in $M/G/1$ and $Geo/G_D/1$ systems, respectively. In *Section 4* we define the way of generating of the output stream of unserved customers during the waiting time. We determine the LST (in continues case) and the p.g.f. (in discrete case) of the length of the interval between two consecutive output moments. We find the mean and the variance of this characteristics. As a consequence, we obtain the LST and p.g.f. of this characteristics when the input stream is ordinary. In *Section 5* we give some conclusions.

2. Description of systems

In this paper we consider systems with arbitrary distributed interarrival and service times which can be continues ($G^X/G/1$ systems) or discrete random variables ($G_D^X/G_D/1$ systems). The input stream is determined by interarrival times $T_1, T_2, \ldots$ which are independent and identically distributed random variables (r.v). In the continues case, they are given by its LST $\alpha(s)$ and in discrete case, by its p.g.f $V(z)$. In each moment, at most one group of customers arrives in the system. The number of customers in a group is a discrete r.v. Z defined by its p.g.f. $\Phi(z)$.

In further, for a given discrete r.v. ξ, we denote its p.g.f by $P_\xi(z)$. If $\xi \geq 0$ is assumed to be a non-integer r.v., the it's LST is denoted by $\varphi_\xi(s)$

In the continues case, if a group arrives in an empty system the service of a customer from this group starts immediately. In the discrete case, the time axis is divided into equal intervals called *slots*. The service of a customer is synchronized to start only at slot boundaries. Without loss of generality, we assume that the length of a slot is equal to a unit time. If a group arrives in an empty system the service of a customer from this group starts in the first discrete moment after arrival epoch.

In the both cases, if the server is busy, the customers from the group remain in the queue and wait for service according to the queueing discipline. We assume that the service time of a customer is given by a positive r.v. X.

3. Some preliminary results

In this section we consider systems with Poisson and geometrical ordinary input stream i.e. $M/G/1$ and $Geo/G_D/1$ queueing systems. (The number of customers in a group is 1).

Let Y be a r.v. which denote the number of customers arrived in the system during the service of a customer. It is a discrete r.v. and its p.g.f. is determined by the following theorem.

Theorem 1. (*see* [3]) *The r.v. Y is determined as follows:*
(a) *for $M/G/1$ system, by the p.g.f.*

$$P_Y(z) = \varphi_X(\lambda - \lambda z), \qquad 0 \le z \le 1$$

(b) (*see* [4]) *for $Geo/G_D/1$ system, by the p.g.f.*

$$P_Y(z) = P_X(q_0 + p_0 z), \qquad 0 \le z \le 1.$$

□

Example 1. (see [3]) If the system $M/M/1$ is considered then the p.g.f. of Y has the form:

$$P_Y(z) = \frac{\mu}{\mu+\lambda}\sum_{k=0}^{\infty}\left(1 - \frac{\mu}{\mu+\lambda}\right)^k z^k \tag{1}$$

and we find that

$$\begin{aligned} P\{Y = 0\} &= \frac{\mu}{\mu+\lambda} \\ P\{Y = k\} &= \frac{\mu}{\mu+\lambda}\left(1 - \frac{\mu}{\mu+\lambda}\right)^k, \quad k = 1,2,\ldots \end{aligned} \tag{2}$$

i.e. $Y \sim Geo(\frac{\mu}{\mu+\lambda})$ or $Y \sim \frac{1}{1+\rho}$, where $\rho = \frac{\lambda}{\mu}$ is the coefficient of occupation of the system.

Example 2. (see [4]) If the system $Geo(p_0)/Geo/1$ is considered then the p.g.f. of Y has the form:

$$P_Y(z) = \tag{3}$$

$$= \frac{bq_0}{1-(1-b)q_0} + \frac{p_0}{1-(1-b)q_0}\sum_{k=1}^{\infty}\left(1 - \frac{b}{1-(1-b)q_0}\right)^{k-1}\frac{b}{1-(1-b)q_0}z^k$$

and the distribution of Y is given as follows:

$$\begin{aligned} P\{Y = 0\} &= P_Y(0) = \frac{bq_0}{1-(1-b)q_0}, \\ P\{Y > 0\} &= 1 - \frac{bq_0}{1-(1-b)q_0} = \frac{p_0}{1-(1-b)q_0}, \\ P\{Y = k\} &= P\{Y > 0\}\frac{b}{1-(1-b)q_0}\left(1 - \frac{b}{1-(1-b)q_0}\right)^{k-1} \quad k = 1,2,\ldots \end{aligned} \tag{4}$$

The last equation implies the follows. Let define the events A and B as

$$\mathbf{A} = \left\{ \begin{array}{c} \text{at least a customer of the input geometrical stream} \\ \text{with parameter } p_0 \text{ arrive in the system during} \\ \text{the service time of a customer} \end{array} \right\}.$$

$$\mathbf{B} = \left\{ \begin{array}{c} \text{exactly } k \text{ customers from a geometrical stream} \\ \text{with parameter } \dfrac{b}{1-(1-b)q_0} \text{ arrive in the system} \end{array} \right\}.$$

Then the probability of the event that exactly k customers from the input stream with parameter p_0, arrive in the system during the service time of a customer is equal to the product of the probabilities of the events **A** and **B**. This means that the r.v. Y has modificated geometrical distribution with the parameter $\dfrac{b}{1-(1-b)q_0}$.

4. A distribution of output stream of unserved customers

At first, let us consider $G^X/G/1$ queueing system (system with arbitrary continuous distribution of interarrival and service times) with the following mechanism of servicing: If a group arrives in an empty system, the service of a customer of this group starts at the same moment when it arrives. On the contrary, the customers of the group remain in the queue and wait for their service. During their waiting times, the customers lose some of their properties and cannot be accepted for servicing. Therefore, after finishing of the service of all customers which belong to the first group, the customers in the queue wait for the arrival of the following group which customers will be accepted for service and in this moment they leave the system unserved.

If $G_D^X/G_D/1$ system is considered, the mechanism of servicing is similar as previous with one difference. Namely, the unserved customers leave the system at the first discrete moment after arrival of the following group which customers will be accepted for servicing.

Let Y_i is a r.v. which denote the number of customers arrived in the system during the service of the i-th customer of a group, $i = 1, \dots, Z$. It is clear that Y_i, $i = 1, \dots, Z$ are i.i.d. random variables. We determine the distribution of the length of the interval between output moments of two consecutive batches of unserved customers. Let denote the length of this interval with U_Z. This interval is composed of a random number $Y_1 + Y_2 + \cdots + Y_Z$ of consecutive interarrival times and one additional interarrival time for the group which will be accepted for servicing. Let T_i be the i-th interarrival time, $i = 1, 2, \ldots$. Namely,

$$U_Z = T_1 + \ldots + T_{Y_1} + T_{Y_1+1} + \ldots + T_{Y_1+Y_2} + \ldots + T_{Y_1+\ldots+Y_Z} + T_{Y_1+\ldots+Y_Z+1}$$

Theorem 2. *The r.v. U_Z is determined as follows:*
(a) *in the continues case, by the LST*

$$\gamma_{U_Z}(s) = \alpha(s)\Phi(P_Y(\alpha(s))) \tag{5}$$

(b) *in the discrete case, by the p.g.f.*

$$P_{U_Z}(z) = V(z)\Phi(P_Y(V(z))) \tag{6}$$

Proof:
(a)

$$\gamma_{U_Z}(s) = E\left(e^{-sU_Z}\right) = E\left(e^{-s\sum_{i=1}^{Y_1+\cdots+Y_Z+1} T_i}\right)$$

$$= \sum_{k=0}^{\infty} E\left(e^{-s\sum_{i=1}^{Y_1+\cdots+Y_k+1} T_i}\right) P\{Z = k\}$$

$$= \sum_{k=0}^{\infty} P\{Z=k\} \sum_{r_1=1}^{\infty} \cdots \sum_{r_k=1}^{\infty} E\left(e^{-s\sum_{i=1}^{r_1+\cdots+r_k+1} T_i}\right) P\{Y_1=r_1\} \cdots P\{Y_k=r_k\}$$

$$= \sum_{k=0}^{\infty} P\{Z = k\} \sum_{r_1=1}^{\infty} P\{Y_1 = r_1\} \cdots \sum_{r_k=1}^{\infty} P\{Y_k = r_k\} \prod_{i=1}^{r_1+\cdots+r_k+1} E\left(e^{-sT}\right)$$

$$= \sum_{k=0}^{\infty} P\{Z = k\} \sum_{r_1=1}^{\infty} P\{Y_1 = r_1\} \cdots \sum_{r_k=1}^{\infty} P\{Y_k = r_k\}(\alpha(s))^{r_1+\cdots+r_k+1}$$

$$= \alpha(s) \sum_{k=0}^{\infty} P\{Z = k\} \sum_{r_1=1}^{\infty} P\{Y_1 = r_1\}\alpha(s)^{r_1} \cdots \sum_{r_k=1}^{\infty} P\{Y_k = r_k\}\alpha(s)^{r_k}$$

$$= \alpha(s) \sum_{k=0}^{\infty} P\{Z = k\} P_{Y_1}(\alpha(s)) \cdots P_{Y_k}(\alpha(s))$$

$$= \alpha(s) \sum_{k=0}^{\infty} P\{Z = k\} \left(P_Y(\alpha(s))\right)^k$$

$$= \alpha(s)\Phi(P_Y(\alpha(s)))$$

(b)

$$P_{U_Z}(z) = \sum_{k=1}^{\infty} P\{U_Z = k\} z^k$$

$$= \sum_{k=1}^{\infty} P\{T_1 + \cdots + T_{Y_1} + \cdots + T_{Y_1+\cdots+Y_Z+1} = k\} z^k$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} P\{T_1 + \cdots + T_{Y_1+\cdots+Y_Z+1} = k | Z = i\} P\{Z = i\} z^k$$

$$= \sum_{i=1}^{\infty} P\{Z = i\} \sum_{k=1}^{\infty} \sum_{r_1=1}^{\infty} \cdots \sum_{r_i=1}^{\infty}$$
$$\times P\{T_1 + \cdots + T_{Y_1+\cdots+Y_i+1} = k \mid Y_1 = r_1, \cdots Y_i = r_i\}$$
$$\times P\{Y_1 = r_1\}, \cdots P\{Y_i = r_i\} z^k$$

$$= \sum_{i=1}^{\infty} P\{Z = i\} \sum_{r_1=1}^{\infty} P\{Y_1 = r_1\} \cdots \sum_{r_i=1}^{\infty} P\{Y_i = r_i\} (V(z))^{r_1+\cdots r_i+1}$$

$$= V(z) \sum_{i=1}^{\infty} P\{Z = i\} \sum_{r_1=1}^{\infty} P\{Y_1 = r_1\} (V(z))^{r_1} \cdots \sum_{r_i=1}^{\infty}$$
$$\times P\{Y_i = r_i\} (V(z))^{r_i}$$

$$= V(z) \sum_{i=1}^{\infty} P\{Z = i\} P_{Y_1}(V(z)) \cdots P_{Y_i}(V(z))$$

$$= V(z) \sum_{i=1}^{\infty} P\{Z = i\} (P_{Y_1}(V(z)))^i$$

$$= V(z)\Phi(P_Y(V(z))).$$

□

Theorem 3. *In the both cases, the mean and the variance of the r.v. U_Z are given by the following expressions:*

$$EU_Z = ET(EY \cdot EZ + 1)$$

$$DU_Z = DT(EY \cdot EZ + 1) + (ET)^2((EY)^2 DZ + EZ \cdot DY).$$

□

Let us consider systems with ordinary input stream i.e. the number of customers in a group is 1. Then the r.v. U_Z is given by

$$U_Z = T_1 + T_2 + \cdots + T_Y + T_{Y+1},$$

where Y denotes the number of customers arrived in the system during the service of a customer.

Corollary 1. *The r.v. U_Z is determined as follows:*
(a) (*see* [2]) *in the continues case, by the LST*

$$\gamma_{U_Z}(s) = \alpha(s) P_Y(\alpha(s)) \tag{7}$$

(b) in the discrete case, by the p.g.f.

$$P_{U_Z}(z) = V(z) P_Y(V(z)). \tag{8}$$

In the both cases, the mean and the variance of the r.v. U_Z is given as follows:

$$EU_Z = ET(EY + 1)$$

$$DU_Z = DT(EY + 1) + DY(ET)^2.$$

□

Example 3. Let us consider the $M/M/1$ queueing system. Using the relation (1) and (7), we have:

$$\gamma_T(s) = \frac{\lambda\mu}{\lambda\mu + (\lambda + \mu)s}$$

and the corresponding density of the distribution of T_Y is given by

$$g_T(t) = \frac{\lambda\mu}{\lambda + \mu} \exp\left(-\frac{\lambda\mu}{\lambda + \mu} t\right), \qquad t \geq 0.$$

So, we have the exponential distribution with the parameter $\dfrac{\lambda\mu}{\lambda + \mu}$. Since the number of customer which leave the system unserved are given by the r.v. Y and for this system its distribution is $Geo\left(\frac{1}{1+\rho}\right)$ (according relation (2)), we obtain that the output stream of unserved customer is quasi-Poisson.

Example 4. Let consider the $Geo(p_0)/Geo(b)/1$ system. In this case, using (3) and (8), we obtain that:

$$P_{T_Y}(z) = \frac{bq_0}{(1-b)p_0+b}\sum_{i=1}^{\infty} q_0^{i-1}p_0 z^i +$$

$$+\frac{p_0}{(1-b)p_0+b}\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}\frac{bp_0^2q_0^{i-1}}{(1-b)p_0+b}\left(1-\frac{bp_0}{(1-b)p_0+b}\right)^{n-i-1} z^n$$

Using the last expression, we find the distribution of the r.v. T_Y as follows.

$$P\{T_Y=1\} = \frac{bp_0q_0^2}{(1-b)p_0+b}$$

$$P\{T_Y=n\} = \frac{bp_0q_0^n}{(1-b)p_0+b}+$$

$$+\frac{p_0}{(1-b)p_0+b}$$

$$\times\sum_{i=1}^{n-1}\frac{bp_0^2q_0^{i-1}}{(1-b)p_0+b}\left(1-\frac{bp_0}{(1-b)p_0+b}\right)^{n-i-1}, \quad n\geq 2.$$

5. Conclusions

In this paper we consider an output stream of groups of unserved customers during the waiting time in $G/G/1$ and $G_D/G_D^X/1$ systems. Namely, during their waiting times the customers lose some of their properties and cannot be accepted for servicing. The distribution of the length of the interval between two consecutive output moments are found. We can notice that the LST (in continues case, (5)) and the p.g.f. (in discrete case, (6)) have the analogues form. Also, the mean and the variance of this characteristics are given by the same expression in the both cases. As a consequence, we obtain the LST and p.g.f. of this characteristic when the input stream is ordinary.

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НЕКОИ КАРАКТЕРИСТИКИ НА ИЗЛЕЗНИТЕ ПОТОЦИ ОД НЕОПСЛУЖЕНИ КЛИЕНТИ ВО $G^X/G/1$ И $G_D^X/G_D/1$ СИСТЕМИ

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Резиме

Во овој труд е проучен еден излезен поток од неопслужени клиенти. Имено, за време на опслужување на еден клиент или на една група клиенти, клиентите кои чекаат во редицата губат некои од своите особини и не можат да бидат прифатени за опслужување. Затоа, тие го напуштаат системот неопслужени. Во трудот е определена Лаплас-Стилтесовата трансформација и генерирачката функција на должината на интервалот помеѓу две последователни излегувања на групи од неопслужени клиенти во $G^X/G/1$ и $G_D^X/G_D/1$ системите, соодветно. Како специјален случај, најдена е распределбата на оваа карактеристика за $M/M/1$ и $Geo/Geo/1$ системите.

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