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DETERMINANT WHOSE ELEMENTS ARE SYMMETRIC FUNCTIONS

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Abstract

In this paper is given one theorem of the "small" generalized Vandermonde determinant.

1. Introduction

In the previous paper [1] we obtained one interesting determinant so called "small" generalized Vandermonde determinant, but here we will give one method of its evaluation. This determinant is important and finds application in the theory of complex manifolds.

In the present paper we denote by σ_s the s-th symmetric function, i.e.

$$\sigma_s(x_1, x_2, \cdots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_s \le n} x_{i_1} \cdot x_{i_2} \cdots x_{i_s} \quad .$$

for $1 \le s \le n$, and $\sigma_0(x_1, x_2, \cdots, x_n) = 1$. Specially, we denote by $\sigma_r(x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n})$ σ_r-th symmetric functions of $r_1 + \cdots + r_n$ variables: x_1 (r_1 times), x_2 (r_2 times), $\cdots$, x_n (r_n times). Also by D we denote differentiation with respect to t.

2. Main result

Now we give the following result.

Theorem. *Let $a_1, \cdots, a_k$ be k distinct numbers and let us consider the following sequence of $m \ge k$ elements*

$$a_1, \cdots, a_1, a_2, \cdots, a_2, \cdots, a_k, \cdots, a_k$$

where a_i repeats r_i times $(1 \leq i \leq k)$. For each $i \in \{1, \cdots, k\}$ we define r_i vector-rows $\mathbf{x}_{i1}, \cdots, \mathbf{x}_{ir_i}$ such that for each $s \in \{1, \cdots, r_i\}$, the j-th $(1 \leq j \leq m)$ coordinate of $\mathbf{x}_{is}$ is

$$\sigma_{j-s}(a_1^{r_1} \cdots a_{i-1}^{r_{i-1}} a_{i+1}^{r_{i+1}} \cdots a_k^{r_k}).$$

Then the determinant of the $m \times m$ matrix obtained from these m vector-rows is equal to

$$(-1)^{m-k} \prod_{1 \leq i < j \leq k} (a_j - a_i)^{r_i r_j}.$$

Proof. Let us consider this "small" generalized Vandermonde determinant as a limit of the standard Vandermonde determinant $V(t)$ with elements

$$a_j + pt, \qquad (1 \leq j \leq k; 0 \leq p \leq r_j - 1) \text{ and } t \neq 0.$$

Let $z(a) = (1, a, a^2, \cdots, a^{n-1})^T$ be a vector-column. If we substitute $z_{jp} = z(a_j + pt)$, where $z_j = z_{j,0}$ and if we introduce the difference operator by the second indices $\Delta z_{jp} = z_{j,p+1} - z_{jp}$, then for the standard Vandermonde determinant we obtain

$$V(t) = det \, \| z_{10}, z_{11}, \cdots, z_{1,r_1-1}, z_{20}, z_{21}, \cdots, z_{2,r_2-1}, \cdots, z_{k0}, z_{k1}, \cdots, z_{k,r_k-1} \| =$$
$$= det \, \| z_{10}, \Delta z_{10}, \cdots, \Delta^{r_1-1} z_{10}, z_{20}, \Delta z_{20}, \cdots, \Delta^{r_2-1} z_{20}, \cdots, z_{k0}, \Delta z_{k0}, \cdots, \Delta^{r_k-1} z_{k0} \|.$$

Let $D^p z_j = D^p z(a_j)$ and $z_j = z(a_j)$, then the "small" generalized Vandermonde determinant V can be expressed in the following way

$$V = det \, \| z_1, \frac{Dz_1}{1!}, \cdots, \frac{D^{r_1-1} z_1}{(r_1-1)!}, z_2, \frac{Dz_2}{1!}, \cdots, \frac{D^{r_2-1} z_2}{(r_2-1)!}, \cdots, z_k, \frac{Dz_k}{1!}, \cdots, \frac{D^{r_k-1} z_k}{(r_k-1)!} \|,$$

where it holds

$$D^p z_j = \lim_{t \to 0} \frac{\Delta^p z_{j0}}{t^p}.$$

Thus after the factorization out of the limits and the scalars, we obtain

$$V = \lim_{t \to 0} \frac{V(t)}{\prod_{j=1}^{k} \prod_{p=0}^{r_j-1} p! t^p}.$$

The final formula

$$V = (-1)^{m-k} \prod_{0 \leq i < j \leq v} (x_j - x_i)^{r_i r_j}$$

follows from the development of the previous limit by using the standard Vandermonde formula for $V(t)$. $\|$

Example. If $m = 6$, $k = 3$, $r_1 = 3$, $r_2 = 2$, $r_3 = 1$, $a_1 = x$, $a_2 = y$ and $a_3 = z$, then according to the theorem it holds

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 2y+z & 1 & 0 & 3x+z & 1 & 3x+2y \\ y^2+2yz & 2y+z & 1 & 3x^2+3xz & 3x+z & 3x^2+y^2+6xy \\ y^2z & y^2+2yz & 2y+z & x^3+3x^2z & 3x^2+3xz & x^3+6x^2y+3xy^2 \\ 0 & y^2z & y^2+2yz & x^3z & x^3+3x^2z & 3x^2y^2+2x^3y \\ 0 & 0 & y^2z & 0 & x^3z & x^3y^2 \end{vmatrix} =$$

$$= -(y-x)^{3\cdot 2}(z-x)^{3\cdot 1}(z-y)^{2\cdot 1} = -(y-x)^6(z-x)^3(z-y)^2.$$

Remark 1. If it is $r_i = 1, (1 \le i \le k)$ then we obtain the standard Vandermonde formula, [2].

Remark 2. This result generalizes as follows. Let

$$a_{11}, a_{12}, \cdots, a_{1,r_1}, a_{21}, a_{22}, \cdots, a_{2,r_2}, \cdots, a_{k1}, a_{k2}, \cdots, a_{k,r_k},$$

be given numbers and let $r_1 + r_2 + \cdots + r_k = m$. For each $i \in \{1, \cdots, k\}$ we define r_i m-dimensional vector-rows $\mathbf{x}_{i1}, \cdots, \mathbf{x}_{ir_i}$ such that for each $s \in \{1, \cdots, r_i\}$, the j-th $(1 \le j \le m)$ coordinate of $\mathbf{x}_{is}$ be the symmetric function

$$\sigma_{j-s}(a_{11}, a_{12}, \cdots, a_{1,r_1}, \cdots, a_{i-1,1}, a_{i-1,2}, \cdots, a_{i-1,r_{i-1}},$$

$$a_{i+1,1}, a_{i+1,2}, \cdots, a_{i+1,r_{i+1}}, \cdots, a_{k1}, a_{k2}, \cdots, a_{k,r_k}).$$

Then the determinant of the $m \times m$ matrix obtained from these m vector-rows is equal to

$$(-1)^{m-k} \prod (a_{jp} - a_{iq}).$$

for $1 \le i < j \le k, 1 \le p \le r_j, 1 \le q \le r_i$.

Note that this result may find some applications in different topics of mathematics. For example it applies in the theory of permutation products on manifolds [1].

References

[1] K.Trenčevski, *Permutation products of 1-dimensional complex manifolds*, Contributions. Sect. Math. Tech. Sci. MASA, **XX 1-2** (1999), 29-37.

[2] B.S.Vatssa, *Theory of Matrices*, 2-nd Ed., Wiley Eastern Ltd., New Delhi 1995.

ЕДНА ДЕТЕРМИНАНТА ЧИИ ЕЛЕМЕНТИ СЕ СИМЕТРИЧНИ ФУНКЦИИ

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Р е з и м е

Во трудот е дадена теорема за вредноста на една детерминанта, којашто е обопштување на Вандермондовата детерминанта.

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