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CONVERGENCE AND INTEGRABILITY OF COSINE TRIGONOMETRIC SUMS WITH QUASI-CONVEX COEFFICIENTS

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Abstract

In this paper we obtain a new proof of theorem for necessary and sufficient condition for the convergence and integrability of cosine trigonometric sums with quasi-convex coefficients.

1. Introduction

Let $\{a_n\}$ be a sequence of real numbers, and we define sequence $\{\Delta a_n\}$ by the formula: $\Delta a_n = a_n - a_{n+1}$, for every $n \in N$.

Thus in view of the above definition it is obvious that the sequence $\{a_n\}$ is monotonely decreasing on N if and only if $\Delta a_n \geq 0$.

Next we define sequence:

$$\Delta^2 a_n = \Delta a_n - \Delta a_{n+1} = a_n - a_{n+1} - (a_{n+1} - a_{n+2}) = a_n - 2a_{n+1} + a_{n+2}, \quad n \in N .$$

Definition 1. A sequence $\{a_n\}$ satisfying $\sum_{n=1}^{\infty} (n+1) |\Delta^2 a_n| < \infty$ is called *quasi-convex.*

Definition 2. A sequence $\{a_n\}$ satisfying $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$ is called *sequence of bounded variation.*

Lemma 1. [1] Let $\{a_n\}$ be quasi–convex and bounded sequence. Then $\{a_n\}$ is sequence of bounded variation, and $\lim\limits_{n\to\infty} n\,\Delta a_n = 0$.

2. Main result

Theorem. Let $\{a_n\}$ be quasi-convex sequence. Then the cosine trigonometric sums is convergent on L^1 to f and holds:

$$\lim_{n\to\infty} \|S_n - f\|_1 = 0 \qquad \text{if and only if} \qquad a_n \log n = o(1)\,.$$

Proof. Applying Abel's transformation twice we get:

$$S_n(x) = \frac{1}{2}\sum_{k=1}^{n} \Delta a_k D_k(x) + \frac{1}{2}a_{n+1}D_n(x) =$$
$$= \frac{1}{2}\sum_{k=0}^{n-2} (n+1)\,\Delta^2 a_n F_n(x) + \frac{1}{2}n\,\Delta a_{n-1}F_{n-1}(x) + \frac{1}{2}a_n D_n(x)$$

where:

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}, \qquad F_n(x) = \frac{1}{n+1}\left[\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\right]^2$$

are Dirichlet and Fejer kernels.

Since $\{a_n\}$ is quasi–convex sequence, it follows that the series $\sum\limits_{n=0}^{\infty}(n+1)\,\Delta^2 a_n \frac{1}{2}F_n(x)$ is convergent in L^1 to f.

Thus

$$\lim_{n\to\infty} \|S_n - f\|_1 = 0$$

if and only if

$$\frac{1}{2}n\,\Delta a_{n-1}F_{n-1}(x) + \frac{1}{2}a_n D_n(x) \to 0 \qquad \text{in} \qquad L^1\ (^*)\,.$$

Sufficiently. Suppose condition $a_n \log n = o(1)$, hold. Then, $a_n = o(1)$. Applying Lemma 1 we have: $n\,\Delta a_n = o(1)$.

Thus the convergence (*) follows by the estimate:

$$\left\|\frac{1}{2}n\,\Delta a_{n-1}F_{n-1}(x) + \frac{1}{2}a_n D_n(x) - 0\right\|_1 \le \frac{1}{2}n\,\Delta a_{n-1}\,\|F_{n-1}\|_1 + \frac{1}{2}a_n\,\|D_n\|_1 =$$
$$= \frac{1}{2}n\,\Delta a_{n-1} + \frac{1}{2}a_n\,\frac{4}{\pi^2}\log n\,.$$

Necessity. Suppose convergence (*) hold. Since

$$\frac{1}{2}N\,\Delta a_{N-1}\hat{F}_{N-1}(n)+\frac{1}{2}a_N\hat{D}_N(n)=$$
$$=\frac{1}{2}N\,\Delta a_{N-1}\int_0^{2\pi}F_{N-1}(x)e^{-inx}\,dx+\frac{1}{2}a_n\int_0^{2\pi}D_N(x)e^{-inx}\,dx=$$
$$=\int_0^{2\pi}\left[\frac{1}{2}N\,\Delta a_{N-1}\left(F_{N-1}(x)e^{-inx}\right)+\frac{1}{2}a_N\left(D_N(x)e^{-inx}\right)\right]dx\,,$$

we have:

$$\lim_{N\to\infty}\left|\frac{1}{2}N\,\Delta a_{N-1}\hat{F}_{N-1}(n)+\frac{1}{2}a_N\hat{D}_N(n)\right|=$$
$$=\lim_{N\to\infty}\left|\int_0^{2\pi}\left[\frac{1}{2}N\,\Delta a_{N-1}\left(F_{N-1}(x)e^{-inx}\right)+\frac{1}{2}a_N\left(D_N(x)e^{-inx}\right)\right]dx\right|\le$$
$$\le\lim_{N\to\infty}\int_0^{2\pi}\left|\left[\frac{1}{2}N\,\Delta a_{N-1}F_{N-1}(x)+\frac{1}{2}a_ND_N(x)\right]e^{-inx}\right|dx=$$
$$=\lim_{N\to\infty}\int_0^{2\pi}\left|\frac{1}{2}N\,\Delta a_{N-1}F_{N-1}(x)+\frac{1}{2}a_ND_N(x)\right|dx=$$
$$=\lim_{N\to\infty}\left\|\frac{1}{2}N\,\Delta a_{N-1}F_{N-1}(x)+\frac{1}{2}a_ND_N(x)\right\|_1=0$$

Now let $n=0$, and then $n=\dfrac{N}{2}$ or $n=\dfrac{N-1}{2}$ if N is even or odd.

Then we get: $N\,\Delta a_{N-1}=o(1)$. Since $\|F_{N-1}\|_1=1$, the convergence (*) imply that $a_ND_N\to 0$ in L^1.

Applying the asymptotic formulae:

$$\|D_n\|_1\sim\frac{4}{\pi^2}\log n+O(1)\qquad\text{we get}:\qquad a_n\log n=o(1)\,.$$

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КОНВЕРГЕНЦИЈА И ИНТЕГРАБИЛНОСТ НА КОСИНУСНИТЕ ТИГОНОМЕТРИСКИ СУМИ СО КВАЗИ–КОНВЕКСНИ КОЕФИЦИЕНТИ

Томовски Живорад

Р е з и м е

Во овој труд е даден нов доказ на тврдењето за потребен и доволен услов за конвергенција и интеграбилност на косинусните тригонометриски суми со квази–конвексни коефициенти.

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