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ON THE DENSITY ON SOME SPECIAL FUNCTIONS IN L^2 SPACE AND IN COMPLEX REGION

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Abstract

In this paper we consider the system of functions $x^n e^{-x/2}$, $n = 1, 2, 3, \dots$ for every $x \in (0, \infty)$. We chall prove that this system of functions is dense in $L^2(0, \infty)$.

Then, let $\lambda_1, \lambda_2, \lambda_3, \dots$ is a sequence of positive intiger numbers, such that $\lambda_n \to \infty$. On the other hand we shall prove that for the system of functions: $z^m e^{1/z}$, $m = 0, 1, 2, 3, \dots$ the sequence $z^{\lambda_1}, z^{\lambda_2}, z^{\lambda_3}, \dots$ is dense, i.e. if for some analytic function $f(z)$ about the point $z = 0$,

$$\int_C f(z) z^{m+\lambda_p} e^{1/z}\, dz = 0\,, \qquad p = 1, 2, 3, \dots \tag{1}$$

where C is a closed curve enclosing $z = 0$ of the complex region Ω and $f \in H(C \cup \operatorname{Int} C)$, then $f(z) = 0$ for every $z \in \Omega$.

We shall prove the following results in main case.

Theorem 1. *The system of functions* $x^n e^{-x/2}$, $n = 1, 2, 3, \dots$ *is dense in* $L^2(0, \infty)$.

Proof. We shall proof that if the function $f \in L^2(0, \infty)$ is an orthogonal by all of functions $x^n e^{-x/2}$, $n = 1, 2, 3, \dots$ then $f = 0$.

Let $\int_0^\infty f(x)\, e^{-x/2}\, x^n\, dx = 0$, and we consider the Laplace's transformation:

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\, e^{-xz}\, dx \qquad \text{for} \qquad z = 1/2 + iy\,.$$

Since $|e^{-xz}| = e^{-x/2}$, we get

$$\int_0^\infty |f(x)|\, e^{-x/2}\, dx \le \left(\int_0^\infty |f(x)|^2\, dx\right)^{1/2} \left(\int_0^\infty e^{-x}\, dx\right)^{1/2} = \|f\|_2 < \infty\,,$$

i.e. the function $f(x)e^{-x/2}$ absolutely convergent for every $x \in (0, \infty)$.

Thus $F(z)$ uniformly convergent at z, for every $x \in (0, \infty)$. Therefore, $F(z)$ is continuous and analytic function.

On the other hand,

$$\frac{\partial^n}{\partial^n}[f(x)\, e^{-xz}] = (-1)^n\, f(x)\, x^n\, e^{-xz}\,, \qquad n = 1, 2, 3, \ldots$$

are continuous functions at z, for every $x \in (0, \infty)$.

Application of the Leibnitz's rule for differentiating under the integral sign, yield:

$$F^{(n)}(z) = \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty f(x)\, x^n\, e^{-xz}\, dx\,, \qquad n = 1, 2, 3, \ldots\,.$$

Specially,

$$F^{(n)}(1/2 + iy) = \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty f(x)\, x^n\, e^{-x/2}\, e^{-ixy}\, dx\,.$$

For $y = 0$ we have:

$$F^{(n)}(1/2) = \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty f(x)\, x^n\, e^{-x/2}\, dx = 0\,, \qquad n = 1, 2, 3, \ldots\,.$$

Applying the Taylor's expansion of function F about the point $z = 1/2$, we get:

$$F(z) = \sum_{n=0}^\infty \frac{F^{(n)}(1/2)}{n} (z - 1/2)^n = 0\,.$$

Thus, $F(z) = \dfrac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\, e^{-x/2}\, e^{iyx}\, dx = 0$. But the uniqueness theorem of Fourier transformation implies that $f(x)\, e^{-x/2} = 0$ i.e. $f(x) = 0$ for every $x \in (0, \infty)$.

Colloraly 1. The set of Lager functions is dense in $L^2(0, \infty)$.

Proof. If $f \in L^2(0, \infty)$ is an orthogonal by all of functions $x^n\, e^{-x/2}$ then f is an orthogonal by all of functions $e^{-x/2}\, L_n(x)$, where $L_n(x) = n!\, \sum_{k=0}^n (-1)^k \binom{n}{k} \dfrac{x^k}{k!}$ are the Lager polynomials.

Using the same technique as in the proof Theorem 1, we get that the system of functions $x^n e^{-x^2}$, $x \in (-\infty, \infty)$ is dense in $L^2(-\infty, \infty)$.

Thus, as a consequence, we get that set of Hermite functions is dense in $L^2(-\infty, \infty)$.

Futher we will define a special class of orthogonal polynomials $P_n(z)$, such that:

$$\int_c e^{1/z} z^m P_n(z) z^v \, dz = 0\,, \qquad v = 1, 2, \ldots, n-1 \tag{2}$$

where c is a closed curve enclosing $z = 0$.

Using calculus for the n–th derivate of the functions $z^{m+2n} e^{1/z}$, $m = 0, 1, 2, \ldots,$ we get that

$$P_n(z) = z^{-m} e^{-1/z} \frac{d^n}{dz^n} \left(z^{m+2n} e^{1/z}\right), \tag{3}$$

is a polynomial of degree n.

Thus $P_n(z) = \lambda_n^{(n)} z^n + \lambda_{n-1}^{(n)} z^{n-1} + \cdots + \lambda_0^{(n)}$, where

$$\lambda_n^{(n)} = (m+2n)(m+2n-1) \ldots (m+n+1),\ \lambda_0^{(n)} = (-1)^n .$$

It's obvious that the polynomial (3) satisfy the condition (2).

Really, by integration by parts, we have:

$$\begin{aligned} I_n &= \int_c z^m e^{1/z} z^v P_n(z)\, dz = \int_c z^v \frac{d^n}{dz^n} \left(z^{m+2n} e^{1/z}\right) dz = \\ &= (-v) \int_c z^{v-1} \frac{d^{n-1}}{dz^{n-1}} \left(z^{m+2n} e^{1/z}\right) dz = \cdots \\ &= (-1)^v v! \int_c \frac{d^{n-v}}{dz^{n-v}} \left(z^{m+2n} e^{1/z}\right) dz . \end{aligned}$$

Now, we consider the integral:

$$g_n = \frac{1}{2\pi i} \int_c z^m e^{1/2} \left(P_n(z)\right)^2 dz .$$

Then, by (2), we get:

$$g_n = \frac{1}{2\pi i} \lambda_n^{(n)} \int_c z^m e^{1/z} P_n(z) z^n \, dz .$$

Application of integration by parts and Cauchy's residue theorem, yield:

$$\int_c z^m\, e^{1/z}\, P_n(z)\, z^n\, dz = \int_c z^n\, \frac{d^n}{dz^n}\,(z^{m+2n}\, e^{1/z})\, dz =$$

$$= (-1)^n\, n!\, \int_c z^{m+2n}\, e^{1/z}\, dz = 2\pi i\, \frac{(-1)^n\, n!}{(m+2n+1)!}\,.$$

Thus,

$$g_n = \frac{(-1)^n\, n!\,(m+2n)(m+2n-1)\cdots(m+n+1)}{(m+2n+1)!} =$$

$$= \frac{(-1)^n\, n!}{(m+2n+1)(m+n)!}\,.$$

Putting $u = z^{m+2n}\, e^{1/z}$, we get that:

$$\frac{u'}{u} = \frac{m+2n}{z} - \frac{1}{z^2}\,, \qquad \text{i.e.} \qquad z^2 u' = u\big[(m+2n)z - 1\big]\,.$$

Thus, differentiating $n+1$–times we find:

$$z^2 u^{(n+2)} + 2\binom{n+1}{1} z u^{(n+1)} + 2\binom{n+1}{2} u^{(n)} =$$

$$= \big[(m+2n)z - 1\big]u^{(n+1)} + \binom{n+1}{1}(m+2n)u^{(n)}\,.$$

If $u^{(n)} = z^m\, e^{1/z}\, y$, then the given differential equation for the polynomials $P_n(z)$, is equivalent to

$$z^2 y'' + \big[(m+2)z - 1\big]y' - n(m+n+1)y = 0\,.$$

The polynomials $P_n(z)$ are usually called generallized Bessel polynomials.

Theorem 2. *The system of functions* $f_m(z) = z^m e^{1/z}$, $m = 0, 1, 2, 3, \ldots$ *is dense in the complex region* Ω.

Proof. Let $f(z)$ is analytic function about the point $z = 0$. Then $f(z) = \sum_{n=0}^{\infty} c_n\, z^n$.

Applyng the Cauchy's residue theorem, we get:

$$\int_c z^{m+\lambda_p+n}\, e^{1/z}\, dz = \frac{2\pi i}{(m+\lambda_p+n+1)!}\,.$$

Then from the condition (1) we have:

$$\sum_{n=0}^{\infty} \frac{c_n}{(m+\lambda_p+n+1)!} = 0\,, \qquad p=1,2,3,\ldots\,.$$

Thus

$$-c_0 = \frac{c_1}{\mu_p+2} + \frac{c_2}{(\mu_p+2)(\mu_p+3)} + \cdots \tag{4}$$

where $\mu_p = m + \lambda_p$.

Then, there exists positive real numbers M and r, such that: $|c_n| < M\,r^n$.

To estimate the coefficient c_0.

$$|c_0| < \frac{M r}{\mu_p+2} + \frac{M\,r^2}{(\mu_p+2)^2} + \frac{M\,r^3}{(\mu_p+2)^3} + \cdots = \frac{M\,r}{\mu_p+2-r} \quad (\lambda_p > r-2-m)\,.$$

Letting $\mu_p \to \infty$, we get $c_0 = 0$. Substituting $c_0 = 0$ in equation (4), we have:

$$-c_1 = \frac{c_2}{\mu_p+3} + \frac{c_3}{(\mu_p+3)(\mu_p+4)} + \cdots\,.$$

In the same way, we obtain $c_1 = 0$, e.t.c. $c_n = 0$ for all $n = 0, 1, 2, \ldots$.

Finally $f(z) = 0$, for every $z \in \Omega$.

In the same way, we have the following corollary:

Corollary 2. The set of Bessel functions is dense in complex region Ω.

Proof. If $f(z) \in \Omega$ is an orthogonal by all of functions $z^m\, e^{1/z}$ then f is an orthogonal by all of functions $z^m\, e^{1/z}\, Q_n(z)$, where $Q_n(z)$ are the Bessel polynomials.

References

[1] D. S. Mitrinović: *Uvod u specijalne funkcije*, Beograd (1991).

[2] M. A. Евграфов: *Аналитические функции*, Москва (1964).

[3] Tomovski Ž.: *On Hermite's and Lager's functions as orthogonal bases in L^2*, Simposium of differential equation, Ohrid (1995).

[4] Walter Rudin: *Real and Complex Analisys*, New York (1976).

ЗА ГУСТИНАТА НА НЕКОИ СПЕЦИЈАЛНИ ФУНКЦИИ ВО L^2 ПРОСТОРОТ И ВО КОМПЛЕКСНА ОБЛАСТ

Томовски Живорад

Р е з и м е

Во овој труд го разгледуваме системот функции: $x^n e^{-x/2}$, $n = 1, 2, 3, \ldots$ за $x \in (0, \infty)$. Докажавме дека овој систем функции е густо множество во Хилбертовиот простор $L^2(0, \infty)$. Понатаму докажавме дека ако $\lambda_1, \lambda_2, \lambda_3, \ldots$ е низа од позитивни цели броеви, т.ш. $\lambda_n \to \infty$, тогаш за системот функции: $z^m e^{1/z}$ низата $z^{\lambda_1}, z^{\lambda_2}, z^{\lambda_3}, \ldots$ е густа, т.е. ако за некоја аналитичка функција $f(z)$ во околина на точката $z = 0$,

$$\int_C f(z) z^{m+\lambda_p} e^{1/z} dz = 0, \qquad p = 1, 2, 3, \ldots \tag{1}$$

каде C е затворена крива во комплексна област Ω и $f \in H(C \cup \operatorname{Int} C)$, тогаш $f(z) = 0$ за секоја $z \in \Omega$.

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