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ON SOME OUTPUT PROCESSES IN THE SYSTEMS M/G/1 AND $M^X/G/1$

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Abstract. In this paper we discuss the output process of the group of unserviced customers generated during the waiting time in the systems M/G/1 and $M^X/G/1$. For some special distributions of the servicing time we have found the distributions of the moments for the batches of customers when they leave the systems, as well as the distributions of the size of the batches in explicit form.

Description of the system

In a single server queueing system we introduce discipline of the queue and mechanism of servicing in the following way: if a customer arrives in an empty system, then he is immediately accepted for servicing while, on the contrary, if the system the servicing which is in process ends. When the servicing of one customer is over, the following customer is accepted for servicing. The group of customers formed during the period of time of servising of one customer is directed to another single server system with the same distribution of servicing time as in the beginning. In that way a bulk-arrival flow of customers is generated. If we presume that the moments of arrival of the group of customers in the new system forms a Poisson flow with parameter λ, then the generated flow is quasi-Poisson defined by the parameter λ and by the generating function $Q(z)$ of the distribution of the number of customers in the group. Under the proposed mechanism, each customer in the first system is accepted for servicing in an empty system.

Distribution of the flow of the leaving moments of not accepted customers

For both systems (the one which the not accepted customers are leaving and the new one they are entering), it is of interest to study the random variable T_Y - the length of the interval of time between the moments of acceptance of two consecutive customers, for the first system, and the length of the interval of time between the moments of arrivals of two consecutive groups of customers, for the second system.

According to the definition, T_Y is composed of a random number Y of consecutive intervals of the flow of customers that arrive (at the end of each interval from which the customers arrived) during the time of servicing of one customer and one additional interval in which arrived the customer to be accepted. So, if τ_i is the length of the interval between the arrivals of the i^{th} and the $(i+1)^{st}$ customer, then

$$T_Y = \tau_1 + \ldots + \tau_Y + \tau_{Y+1} \tag{1}$$

Let us presume that the arriving flow is recurrent with the distribution function $A(x)=P(\tau_i < x)$, Laplace-Stiltjes transform $\alpha(s)$ and the moments α_1, α_2. Then the Laplace-Stiltjes transform of the distribution of T_Y is defined by

$$\gamma_T(s) = \alpha(s)\cdot Q(\alpha(s)) \tag{2}$$

The mean and the variance of T_Y are given by

$$\begin{aligned} ET_Y &= \alpha_1(1+EY), \\ DT_Y &= \alpha_A^2(1+EY) + \alpha_1^2\cdot DY, \qquad \alpha_A^2 = \alpha_2-\alpha_1^2 \end{aligned} \tag{3}$$

The Systems M/G/1

For the systems of the type M/G/1, the random variable Y-the length of the queue generated during the period of time of one servicing, the generating function is defined, see [2], by

$$Q(z) = \beta(\lambda-\lambda z), \quad 0 \le z \le 1,$$

so that the Laplace-Stiltjes transform of T_Y is given by

$$\gamma_T(s) = \alpha(s)\cdot\beta(\lambda-\lambda\alpha(s))$$

If in the last formula we substitute $\alpha(s)$ by the Laplace-Stiltjes transform of the exponential distribution of the intervals τ_i, we get

$$\gamma_T(s) = \frac{\lambda}{\lambda+s}\cdot\beta\left(\frac{\lambda s}{\lambda+s}\right)$$

In [2] is found the distribution of Y for different kinds of distributions of the serving time which enables us to find the Laplace-Stiltjes transform $\gamma_T(s)$ for those cases.

For the systems M/M/1 we have that the Laplace-Stiltjes transform is

$$\gamma(s) = \frac{\lambda\mu}{\lambda\mu+(\lambda+\mu)s},$$

and the corresponding original, i.e. the density of the distribution of T_Y is given by

$$G'(t) = \frac{\lambda\mu}{\lambda+\mu}\exp(-\frac{\lambda\mu}{\lambda+\mu}t), \quad t \geq 0$$

So, we got the exponential distribution with the parameter

$$\hat{\lambda} = \lambda\mu/(\lambda+\mu)$$

It is shown in [2] that Y has geometric distribution

$$P\{Y=k\} = \frac{1}{1+\rho}(1-\frac{1}{1+\rho})^k, \quad \rho = \lambda/\mu, \quad k=0,1,2,\dots ,$$

with $\hat{p} = 1/(1+\rho)$, which in combination with the exponential distribution of T_Y defines the new generated flow of customers to be quasi-Poisson.

If X_t denotes the random variable - number of customers arriving at the system during the period of time t, and $P_k(t) = P\{X_t=k\}$, $k=0,1,2,\dots$, $t > 0$, then the generating function of the newly formed quasi-Poisson flow,

$$\hat{P}(z,t) = \sum_{k=o}^{\infty} \hat{P}_k(t)z^k$$

according to [5], is defined by

$$\hat{P}(z,t) = \exp(-\hat{\lambda}(1-\Phi_1(z))t),$$

where $\Phi_1(z)$ is the generating function for the number of customers in group with geometric distribution $P\{Y=k\} = \hat{p}\hat{q}^k$, $k=0,1,2,\dots$.

From

$$\hat{P}(z,t) = \exp(-\lambda(1-\frac{\hat{p}}{1-\hat{q}z})t) \qquad (4)$$

we immediately get

$$\hat{P}_o(t) = \hat{P}(0,t) = \exp(-\lambda qt)$$

By taking logarithm in (4) and consecutive differentiating (similarly as in [3]), we get

$$\hat{P}_1(t) = \frac{\hat{\lambda}\hat{q}t}{1!}\,\hat{p}\,\exp(-\hat{\lambda}\hat{q}t),$$

$$\hat{P}_2(t) = \left(\frac{(\hat{\lambda}\hat{q}t)^2}{2!}\,\hat{p}^2 + \frac{\hat{\lambda}\hat{q}t}{1!}\hat{p}\hat{q}\right)\exp(-\hat{\lambda}\hat{q}t),$$

$$\hat{P}_3(t) = \left(\frac{(\hat{\lambda}\hat{q}t)^3}{3!}\,\hat{p}^3 + \frac{(\hat{\lambda}\hat{q}t)^2}{2!}\,2\hat{p}^2\hat{q} + \frac{\hat{\lambda}\hat{q}t}{1!}\hat{p}\hat{q}^2\right)\exp(-\hat{\lambda}\hat{q}t),$$

and so on. By induction we can conclude that the distribution of X_t is

$$\hat{P}_n(t) = \left(\frac{(\hat{\lambda}\hat{q}t)^n}{n!}\,\hat{p}^n + \frac{(\hat{\lambda}\hat{q}t)^{n-1}}{(n-1)!}\binom{n-1}{1}\hat{p}^{n-1}\hat{q} + \frac{(\hat{\lambda}\hat{q}t)^{n-2}}{(n-2)!}\binom{n-2}{2}\hat{p}^{n-2}\hat{q}^2 + \ldots\right.$$
$$\left.\ldots + \frac{(\hat{\lambda}\hat{q}t)^{n-k}}{(n-k)!}\binom{n-k}{k}\hat{p}^{n-k}\hat{q}^k + \ldots + \frac{\hat{\lambda}\hat{q}t}{1!}\,\hat{p}\hat{q}^{n-1}\right)\exp(-\hat{\lambda}\hat{q}t),$$

$$\hat{P}_n(t) = \sum_{k=1}^{n}\frac{(\hat{\lambda}\hat{q}t)^k}{k!}\binom{n-1}{k-1}\hat{p}^k\hat{q}^{n-k}\exp(-\hat{\lambda}\hat{q}t), \quad n=1,2,\ldots$$

$$\hat{P}_o(t) = \exp(-\hat{\lambda}\hat{q}t)$$

The validity of the above formulae can be established, as well, by taking into account the probabilistic meaning of $\hat{P}_n(t)$. Namely, $\hat{P}_o\{X_t=0\}$ is the probability that for the interval of time of length t there is no customer from the flow under consideration, which will happen if there is none moment of the Poisson flow with parameter $\hat{\lambda}$ or, there are k moments, k=1,2,..., of the same flow in every of which arrived an "empty" group. So,

$$P\{X_t=0\} = \sum_{k=o}^{\infty}\hat{p}^k\frac{(\hat{\lambda}\hat{t})^k}{k!}\exp(-\hat{\lambda}t) = \exp(-\hat{\lambda}\hat{q}t),$$

which, on the other hand, is the probability that no customer arrived from the Poisson flow with parameter $\hat{\lambda}\hat{q}$. Because of that, now, $\hat{P}\{Y=n\}$ will be the probability of the event: during the period of time t, there has arrived n customers in k moments, k=1,2,...,n, from the Poisson flow with parameter $\hat{\lambda}\hat{q}$, and all groups of customers are nonempty.

If we express $\hat{\lambda}$ and $\hat{p}$ trough the parameters of the first system, we get for $\hat{P}_o(t)$ and $\hat{P}_n(t)$ the following formulae

$$\hat{P}_o(t) = \exp\left(-\frac{\mu\lambda^2}{(\mu+\lambda)^2}t\right),$$

$$\hat{P}_n(t) = \sum_{k=1}^{n}\left(\frac{\mu\lambda^2}{(\mu+\lambda)^2}\right)^k\frac{1}{k!}\binom{n-1}{k-1}\frac{\mu^k\lambda^{n-k}}{(\mu+\lambda)^n}\hat{P}_o(t), \quad n=1,2,\ldots$$

For the systems $M/E_{k-1}/1$ it is show in [2] that the distribution of the number of customers in groups of the leaving flow of unserviced customers is

$$P\{Y=n\} = \binom{k+n-1}{k-1}\frac{1}{(1+\rho)^k}\cdot\frac{\rho^n}{(1+\rho)^n}, \quad \rho = \frac{\lambda}{\mu}, \quad n=0,1,2,\ldots$$

If we put $p_1=1/(1+\rho)$, then the distribution of Y can be interpreted as the distribution of the number of "unsuccessful" experiments before k "successes" happen (with the probability of "success" equal to p_1).

The Laplace-Stiltjes transform of Y in the system $M/E_{k-1}/1$ will be

$$\gamma(s) = \lambda p_1^k \sum_{r=0}^{k-1}\binom{k-1}{r}\frac{(\lambda q_1)^r}{(s+\lambda p_1)^{r+1}}$$

from where, for the density of the distribution, we get

$$G'(t) = \lambda p_1^k \exp(-\lambda p_1^t)\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{(\lambda q_1)^r}{r!}t^r$$

and, after expressing p_1 through λ and μ of the system $M/E_{k-1}/1$, we will have that

$$G'(t) = \sum_{r=0}^{k-1}\binom{k-1}{r}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{k-1-r}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^{r+1}\frac{t^r}{r!}\exp\left(-\frac{\lambda\mu}{\lambda+\mu}t\right)$$

For the mean and the variance we have that

$$E(T_Y) = \frac{1}{\lambda}\left(1+k\frac{1}{p_1}\right) = \frac{1}{\lambda}(1+k\rho_o),$$

$$D(T_Y) = \frac{1}{\lambda^2}\left(1+k\frac{q_1}{p_1}\left(1+\frac{1}{p_1}\right)\right) = \frac{1}{\lambda^2}(1+k\rho_o(2+\rho_o)), \quad \rho_o = \frac{\lambda}{\mu}$$

For the system M/D/1 it is shown in [2] that the variable Y has Poisson distribution with parameter $\rho=\lambda\beta_1$, where β_1 is the mean of the servicing time. If we put $\beta_1=1/\mu$, then we will have that

$$\gamma(s) = \frac{\lambda}{\lambda+s}\exp\left(-\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\lambda+s}\right)\right)$$

By inversion of the Laplace-Stiltjes transform $\gamma(s)$ we get

$$G'(t) = \lambda\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k\cdot\frac{1}{k!}\exp\left(-\frac{\lambda}{\mu}\right)\cdot\frac{(\lambda t)^k}{k!}\exp(-\lambda t)$$

If we denote

$$P_{\lambda/\mu}(k) = \left(\frac{\lambda}{\mu}\right)^k\cdot\frac{1}{k!}\exp\left(-\frac{\lambda}{\mu}\right),$$

$$E'_{k,\lambda} = \frac{\lambda^{k+1} t^k}{k!} \exp(-\lambda t)$$

then the distribution function of Y can be expressed through the Poisson and Erlang distributions as follows

$$G(t) = \sum_{k=0}^{\infty} P_{\lambda/\mu}(k) \cdot E_{k,\lambda}(t)$$

The mean and the variance, in this case, will be

$$ET_Y = \frac{1}{\lambda}(1+\frac{\lambda}{\mu}), \quad DT_Y = \frac{1}{\lambda^2}(1+\frac{2\lambda}{\mu}).$$

Conclusion

The examination of the flow of unserviced customers for the system M/G/1 that we have done above, shows that only in the case of exponential distribution of servicing time we can get the geometric distribution of the size of group and the exponential distribution of the intervals of leaving the system of two consecutive groups.

Systems $M^X/M/1$

Now, let us consider an one-server system with exponentially distributed servicing time and flow of groups of customers that arrive in moments which form a Poisson simple flow.

Let A(t) and $\Phi(z)$ denote the distribution function of the flow of moments of arrivals of the group of customer and the generating function of the size X of the groups, respectively and let B(t) denotes the distribution function of the servicing time. Then the distribution function of the random variable Y* - size of the groups of customers formed during the servicing time of one customer, will be defined by the generating function

$$Q(z) = \beta(\lambda-\lambda\Phi(z)), \quad 0 \le z \le 1$$

where λ is the parameter in the Poisson flow and $\beta(s)$ the Laplace-Stiltjes transform of B(t). If there exist β_1 and β_2 and EX and EX^2, as well, then the first two moments and the variance of Y* will be

$$ET_Y^* = \alpha_1 + \beta_1\Phi'(1),$$

$$ET_Y^{*2} = \alpha_2 + 2\alpha_1\beta_1\Phi'(1) + \alpha_1\beta_1\Phi''(1) + \beta_2\Phi'(1)^2 + \lambda\alpha_2\beta_1\Phi'(1),$$

$$DT_Y^* = \sigma_A^2(1+\lambda\beta_1\Phi'(1)) + \Phi'^2(1)\cdot\sigma_B^2 + \alpha_1\beta_1(\Phi''(1)+\Phi'(1)),$$

$$\sigma_A^2 = \alpha_2 - \alpha_1^2, \qquad \sigma_B^2 = \beta_2 - \beta_1^2$$

If the group of customers waiting to be serviced, leaves the system in the moment when, after the end of servicing of one a new customer arrives, then the Laplace-Stiltjes transform of the interval of time T_Y^* between the moments of leaving the system of two consecutive groups, will be

$$\gamma(s) = \alpha(s)\cdot\beta(\lambda-\lambda\phi(\alpha(s)))$$

or, expressed through the distribution of Y*,

$$\gamma(s) = \alpha(s).Q(\alpha(s))$$

From the last formula we get the mean and the variance of T_Y^*:

$$ET_Y^* = \alpha_1(1+EY^*)$$

$$DT_Y^* = \sigma_A^2(1+EY^*) + \alpha_1^2.DY^*$$

For the system $M^X/M/1$ defined by $\Phi_1(z)=pz/(1-qz)$ and parameter μ of the distribution of servicing time, we have that

$$\gamma(s) = \frac{\lambda\mu}{p\lambda\mu+(\mu+\lambda)s} - \frac{q\lambda^2\mu}{(s+\lambda)[p\lambda\mu+(\mu+\lambda)s]}.$$

By inversion of $\gamma(s)$ we get the density of T_Y^* in the following form

$$G'(t) = \frac{q\lambda\mu}{\lambda+q\mu}\exp(-\lambda t) + \frac{p\lambda^2\mu}{(\lambda+\mu)(\lambda+q\mu)}\exp(-\frac{p\lambda\mu}{\lambda+\mu}t),$$

and the corresponding distribution function

$$G(t) = 1 - \frac{1}{\lambda+q\mu}(q\mu\cdot\exp(-\lambda t)+\lambda\cdot\exp(-\frac{p\lambda\mu}{\lambda+\mu}t)).$$

The mean and the variance of T_Y^* are

$$ET_Y^* = \frac{1}{\lambda} + \frac{1}{p\mu}$$

$$DT_Y^* = (\frac{1}{\lambda})^2 + (\frac{1}{p})^2\cdot((\frac{1}{\mu})^2 + \frac{2}{\lambda\mu}).$$

So, we got that the average length of the interval between two moments of the beginnings of new servicing, i.e. between two consecutive moments of leaving by the groups of the new flow from the system under consideration is for $q/p\mu$ bigger than that of the system M/M/1.

If we turn, now, to the system $M^X/M/1$, defined by the function $\Phi_2(z)=p/(1-qz)$ and parameter μ of the servicing time, then

$$\gamma(s) = \frac{\lambda\mu}{p\lambda\mu+(q\lambda+\mu)s} - \frac{q\lambda^2\mu}{(\lambda+s)[p\lambda\mu+(q\lambda+\mu)s]}$$

from where we get

$$G'(t) = \frac{\lambda\mu}{\lambda+\mu}\exp(-\lambda t) + \frac{p\lambda^2\mu}{(\lambda+\mu)(q\lambda+\mu)}\exp(-\frac{p\lambda\mu}{q\lambda+\mu}t)$$

and the corresponding distribution function

$$G(t) = 1 - \frac{1}{\lambda+\mu}(\mu\cdot\exp(-\lambda t)+\lambda\cdot\exp(-\frac{p\lambda\mu}{q\lambda+\mu}t)).$$

In this case, for the mean and the variance, we have that

$$ET_Y^* = \frac{1}{\lambda} + \frac{q}{p\mu}$$
$$DT_Y^* = (\frac{1}{\lambda})^2 + (\frac{1}{p})^2\cdot((\frac{q}{\mu})^2 + \frac{2q}{\lambda\mu}).$$

We note that the average of the interval between two consecutive moments when the groups are leaving the system, defined by $\Phi_2(z)$ is for $1/\mu$ smaller then the one for the system defined by $\Phi_1(z)$.

If we compare the numerical characteristics of T, we have found for the last case and those for the systems M/M/1, we can conclude that:

-if $p > q$, then $ET_Y^* > ET_Y$;

-if $p < q$, then $ET_Y^* < ET_Y$;

-if $p = q$, then $ET_Y^* = ET_Y$.

Conclusion

From the above discussion we can conclude that for the systems $M^X/G/1$, under the presumption that the servicing is exponential, the output process is unordinary with a kind of geometric distribution for the batch size, but it is not quasi-Poisson. So, if we wish the output process from the system $M^X/M/1$, when entering a new system, to be quasi-Poisson, it is necessary that the arrival batch moments form a simple Poisson flow with parameter λ_1.

R E F E R E N C E S

[1] Георгиева, М.: Карактеристики на една система за масовно опслужување при групно пристигнување на пријавите, Год. зобр., Матем. фак., 28, 1977, 95-97

[2] Георгиева, М.: Распределба на некои катактеристики на системите за масовно опслужување од типот M/G/1 и M^x/G/1, Год. зобр., Матем. фак., 33-34, 1982-1983, 69-76

[3] Георгиева, М.: За еден неординарен влезен поток, Матем. бил., кн. 9-10 (XXXV-XXXVI), 1985-1986, 33-38

[4] Турсунов, Г.Т., Турсунов, Р.Т.: Пределные теоремы для некоторых процесов системы массового обслуживания M/G/1 с груповим поступлением. Пределние теоремы для вероятност. распределении, Ташкент, 1985, 160-172

[5] Обретенов, А., Димитров, Б., Даниелян, Э.: Массово обслужване и приоритетни системи на опслужване, София, 1973

[6] Kleinrock, L.: Queueing systems, John Wiley & Sons, New York, 1975

[7] Деч, Г.: Руководство к практическому применению преобразования Лапласа и Z преобразования, Москва, 1971

ЗА НЕКОИ ИЗЛЕЗНИ ПОТОЦИ КАЈ СИСТЕМИТЕ M/G/1 И M^x/G/1

М. Георгиева

Р е з и м е

Изучени се излезни потоци кај системите од тип M/G/1 и M^x/G/1, формирани од клиентите што пристигнуваат за време на опслужувањето на еден клиент. Така добиените потоци, во општ случај, се со пристигнување во групи. Врз основа на добиените во [2] распределби на обемот Y на групите, тука се наоѓа распределбата на времето T_Y на излегување на групите клиенти, за различни распределби на времето на опслужување, како и за два вида на геометриска распределба на обемот на групите во системите M^x/G/1.