Мат. Бил. 7-8 (XXXIII-XXXIV),
1983-1984, (1-3),
Скопје, Југославија

ON A GENERALIZATION OF THE CHONG KONG MING THEOREM

Josip E. Pečarić

0. Chong Kong Ming ([1]) has proved the following theorem:

THEOREM A. Let $p=(p_1,\dots,p_n)$ and $x=(x_1,\dots,x_n)$ be the nonnegative n-tuples and let function F be defined by

$$F(\lambda) = \left(\prod_{i=1}^{n}\left(\frac{\lambda}{P_n}\sum_{j=1}^{n} p_j x_j + (1-\lambda)x_i\right)^{p_i}\right)^{1/P_n} \quad \left(P_n=\sum_{i=1}^{n} p_i\right). \tag{1}$$

Then $F(\lambda)$ is a nondecreasing function on $[0,1]$.

From Theorem A, it is obtained:

THEOREM B. Let p and x be the nonnegative n-tples and let $0\le a\le b\le 1$. Then

$$G_n(x;p) \le F(a) \le F(b) \le A_n(x;p), \tag{2}$$

where F is defined by (1) and

$$G_n(x;p) = \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}, \quad A_n(x;p) = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i .$$

1. In this paper we shall give the generalizations of Theorems A and B.

THEOREM 1. Let p and x be the nonnegative n-tuples and let the function ϕ be defined by

$$\phi(\lambda,a)=\left(\prod_{i=1}^{n} (\lambda a+(1-\lambda)x_i)^{p_i}\right)^{1/P_n} \Big/ \frac{1}{P_n}\sum_{i=1}^{n} p_i(\lambda a+(1-\lambda)x_i)). \tag{3}$$

Then $\phi(\lambda,a)$ is a nondecreasing function for $0\le\lambda\le 1$ and $0\le a<+\infty$.

PROOF. We shall use the following notations:

$$\phi_a'(\lambda,a) = \frac{\partial\phi(\lambda,a)}{\partial a}, \quad \phi_\lambda'(\lambda,a) = \frac{\partial\phi(\lambda,a)}{\partial \lambda}, \quad \bar{x} = A_n(x;p).$$

From (3) we have

$$\log\phi(\lambda,a) = \frac{1}{P_n}\sum_{i=1}^{n} p_i \log(\lambda a+(1-\lambda)x_i) - \log(\lambda a+(1-\lambda)\bar{x}),$$

so,

$$\frac{\phi'_\lambda(\lambda,a)}{\phi(\lambda,a)} = \frac{1}{P_n}\sum_{i=1}^{n} p_i \frac{a-x_i}{\lambda a+(1-\lambda)x_i} - \frac{a-\bar{x}}{a\lambda+(1-\lambda)\bar{x}},$$

The function $u(x)=(a-x)/(\lambda a+(1-\lambda)x)$ is convex for $x\geq 0$, because $u''(x)=2a(1-\lambda)/(\lambda a+(1-\lambda)x)^3 \geq 0$. Using the Jensen inequality for convex function u,

$$u\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i u(x_i), \tag{4}$$

we have

$$\frac{a - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i}{\lambda a+(1-\lambda)\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i} \leq \sum_{i=1}^{n} p_i \frac{a-x_i}{\lambda a+(1-\lambda)x_i}$$

i.e.

$$\phi'_\lambda(\lambda,a)/\phi(\lambda,a) \geq 0.$$

So, ϕ is a nondecreasing function on $0\leq\lambda\leq 1$ for every $a\in[0,\infty)$.

On the other hand we have

$$\frac{\phi'_\lambda(\lambda,a)}{\phi(\lambda,a)} = \frac{1}{P_n}\sum_{i=1}^{n} p_i \frac{\lambda}{\lambda a+(1-\lambda)x_i} - \frac{\lambda}{\lambda a+(1-\lambda)\bar{x}}.$$

The function $V(x) = \lambda/(\lambda a+(1-\lambda)x)$ is convex for $x\geq 0$, because $V''(x) = 2\lambda(1-\lambda)^2/(\lambda a+(1-\lambda)x)^3 \geq 0$. Using (4), we have

$$\lambda/(\lambda a+(1-\lambda)\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \lambda/(\lambda a+(1-\lambda)x_i),$$

i.e.

$$\phi'_a(\lambda,a)/\phi(\lambda,a) \geq 0.$$

So, ϕ is a nondecreasing function for $0\leq a<+\infty$, too.

Thus the proof is finished.

REMARK: For $a=\bar{x}$, from Theorem 1, we obtain Theorem A.

COROLLARY 2. Let p and x be the nonnegative n-tuples.

If $0 \leq a \leq x \leq b < \infty$ and $0 \leq p \leq q \leq 1$, then

$$A_n(x;p)/G_n(x;p) \leq \phi(p,a) \leq F(p) \leq F(q) \leq \phi(q,b) \leq 1,$$

where F and ϕ are given by (1) and (3).

If $0 \leq a \leq b < \infty$ and $0 \leq p \leq q \leq 1$, then

$$A_n(x;p)/G_n(x;p) \leq \phi(p,a) \leq \begin{Bmatrix} \phi(p,b) \\ \phi(q,a) \end{Bmatrix} \leq \phi(q,b) \leq 1.$$

REFERENCE

[1] CHONG KONG MING: On the aritmetic-mean-geometric mean inequality. Nanta Math. 10, No 1, (1977), 26-27.

Faculty of Civil Engineering
Bulevar Revolucije 73
11 000 Beograd
Yugoslavia