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CONCEPTUAL INTRODUCTION TO UNIQUE-VALUED SEQUENCES

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Abstract. For the first time in this paper is introduced the new concept of the unique-valued sequences, in the following way:

Let $G()$ be a multiplicatively denoted semigroup. For a given positive integer n, let be given a finite (ordered) sequence u_1, u_2,… u_n (where $u_i \in G$, for all i = 1, 2,… n). This sequence is to be called ***unique-valued sequence*** of ***length*** n having its ***unique value*** v, provided there holds: $v = u_1.u_2..\ u_n$

and simultaneously $v \neq u_{\pi(1)}..u_{\pi(2)}..\ .\ u_{\pi(n)}$ for any non-identical permutation π of the set of indices $\{1, 2,\dots n\}$.

Then, this new concept is empirically examined on two suitable families of semigroups, the semigroups of unit matrices of order n and the groups of permutation matrices of order n.

1. Definition of the unique-valued sequences

Although the books (or articles) on topics from the modern algebra are overwhelmingly filled with various concepts, still there might be useful simple concepts waiting to be defined. In this paper, the authors introduce the concept of ***unique-valued sequences***, which offers a new insight into the study of non-commutative algebraic structures. Although this concept seems to be very plain and simple, the heavy investigation of many algebraic books and articles (including the resources provided by the OEIS [2]) suggests that this simple concept has not yet been introduced.

Definition 1. Let $G()$ be a multiplicatively denoted semigroup. For a given positive integer n, let be given a finite (ordered) sequence u_1, u_2,… u_n (where $u_i \in G$, for all indices i = 1, 2,… n). This sequence is to be called ***unique-valued sequence*** of ***length*** n having its ***unique value*** v, provided there holds:

$$v = u_1.u_2..\ u_n \tag{1}$$

and simultaneously

$$v \neq u_{\pi(1)} u_{\pi(2)} \dots u_{\pi(n)}$$

for any non-identical permutation π of the set of indices $\{1, 2,\dots n\}$.

The equation (1), the left hand side of which is the unique value, while the right hand side contains all the elements of the unique-valued sequence multiplied in their initial order, is said to ***present*** the given unique-valued sequence.

Simply said, the definition of a unique-valued sequence means that only way to obtain the unique value v from the elements $u_1, u_2,\dots u_n \in G$ is by multiplying them *in that particular order*; while the other possible orders of multiplying must always give another value, different from v. Of course, the semigroup $G()$ can possess richer structure: it might be a monoid $G(\cdot,e)$, a group $G(\cdot,^{-1},e)$, or a monoid algebra $G(\cdot,+,e)$. The presentation of a unique-valued sequence with an equation such as (1) is made due to practical reason (on the left hand side we can find the unique value, while the right hand side contains all the elements of the unique-valued sequence "separated" by the operation symbol . instead of a comma; and the multiplication of these elements in the given order yields the unique value, granted by the equality sign).

Example 1. Let $\mathbf{S_3}(\circ,\ ^{-1},\ e)$ be the symmetric group of permutations of the three element set {1, 2, 3}, denoted with cycles: $\mathbf{S_3}$ = {(1), (12), (13), (2,3), (123), (132)}. Then the sequence (12), (132), (13) is unique-valued sequence of length 3 with unique value (132), because:

$$(132) = (12)\circ(132)\circ(13) \quad (2)$$

and simultaneously

$$(132) \neq (1) = (12)\circ(13)\circ(132)$$
$$(132) \neq (1) = (132)\circ(12)\circ(13)$$
$$(132) \neq (123) = (132)\circ(13)\circ(12)$$
$$(132) \neq (123) = (13)\circ(12)\circ(13)$$
$$(132) \neq (1) = (13)\circ(132)\circ(12)$$

This unique-valued sequence is presented by the equation (2). ■

For any semigroup *G*(), all the elements $u \in G$ can be observed as trivial unique-valued sequences of length 1. The sequence u_1, u_2 of length 2 is unique-valued if and only if the two elements don't commute with respect to . (then it's value is $v = u_1 \cdot u_2 \neq u_2 \cdot u_1$). There is a simple way for obtaining new (shorter) unique-valued sequences from another (longer) unique-valued sequence, simply by **deleting** some elements from the *beginning* or from the *end* of the given (longer) sequence; the validity of this procedure of "shortening" can be shown by contraposition – if some of the new (shorter) sequences were not unique-valued, neither would be the given (longer) sequence.

Another obvious fact is that **a unique-valued sequence cannot contain two identical elements**, as the *transposition* switching their indices would be a non-identical permutation which produces the same value. However, it is possible for the unique value to be equal to some element of the unique-valued sequence – in **Example 1**, the unique value (132) equals to the second element of the given unique-valued sequence.

2. Properties of the unique-valued sequences

This was the easy part of the story. The complications begin once we ask ourselves: *is there an **upper limit of the length** of the unique-valued sequences, in certain algebraic structures*? The rise of this question *makes sense*, as if one proves that in the semigroup *G*() there is no unique-valued sequence of length *m*, there cannot be unique-valued sequences of all lengths *n*>*m* (if there were some, by "shortening" one of them one could obtain a unique-valued sequence of length *m*, an obvious contradiction to the assumption). For certain types of semigroups this answer is trivial, as it is seen in our first proposition.

Proposition 1. About the existence of an upper limit of the length of the unique-valued sequences we have:

(*i*) if *G*() is a commutative semigroup, then the longest unique-valued sequence has just one element in length;

(*ii*) if $G()$ is free semigroup (or free group) with at least two generators, then there exists an unique-valued sequence of arbitrary length, i.e. there is no upper limit of the length of the unique-valued sequences;
(*iii*) in the finite semigroup $G()$ containing n elements, the longest unique-valued sequence has length no more than n.

Proof.
(*i*) Trivially true, as any two elements from G commute with respect to ., providing there can't be a unique-valued sequence of length 2 (nor longer);
(*ii*) If a and b are two generators, then for an arbitrary positive integer n the following sequence of length n: $ab, ab^2, ab^3, \ldots, ab^n$ is obviously unique-valued with value $v = abab^2ab^3\ldots ab^n$, since its value cannot be equal to another word in the semigroup (or group) G which is free;
(*iii*) If there was a unique-valued sequence of length $n+1$, then by *pigeon hole principle*, it must contain two identical elements, which has already been discussed as impossible (the *transposition* switching their indices produces the same value). ■

The result (*iii*) of the above proposition guarantees that in finite semigroups there is always a maximal length of all possible unique-valued sequences. However, in the several finite semigroups that were empirically researched by the authors of this paper, the length of the longest unique-valued sequence was *fairly lower than* $n=|G|$; so there is significant interest to provide the exact answer of this question for a particular semigroup.

If we have found one unique-valued sequence of length n in the semigroup $G()$, is there some universal way to produce more unique-valued sequences of the same length? Obvious tools to do this are the available homomorphisms, which preserve the structure of the semigroup. However, if a certain homomorphism is not injective, there is no guarantee that the image of a unique-valued sequence would again be a unique-valued sequence (for example, such a homomorphism may map two different elements from the original unique-valued sequence into same element from G, thus producing a non-unique-valued sequence). Fortunately, this doesn't happen when the given homomorphism is injective, i.e. a monomorphism.

Proposition 2. Let $u_1, u_2, \ldots u_n$ be an unique-valued sequence in the semigroup $G()$. If $h: G \to G$ is a monomorphism, then its image $h(u_1), h(u_2), \ldots h(u_n)$ is a unique-valued sequence again.

Proof. Let v be the value of the given unique-valued sequence; then it may be presented by the equation:

$$v = u_1 . u_2 . \ldots . u_n , \tag{1}$$

As h is a homomorphism, the image of the given unique-valued sequence has value:

$$h(v) = h(u_1) . h(u_2) . \ldots . h(u_n) . \tag{2}$$

We now have to prove that $h(u_1), h(u_2), \ldots h(u_n)$ is a unique-valued sequence presented by (2). By contraposition, suppose that there is a (potentially non-identical) permutation Π on the set $H = \{h(u_1), h(u_2), \ldots h(u_n)\}$ such that:

$$h(v) = \Pi(h(u_1)) . \Pi(h(u_2)) . \ldots . \Pi(h(u_n)) . \tag{3}$$

Now, let's define a permutation π of the set of indices $I = \{1, 2, \ldots n\}$ in the following way:

$$\pi(i) = j, \quad \text{if } \Pi(h(u_i)) = h(u_j). \tag{4}$$

First we need to show that π is a well defined mapping. For a given index i, there is a unique element $h(u_i)\in H$. With the permutation Π of H it is mapped to an unique element $\Pi(h(u_i)) = h(u_j)\in H$. Since h is injective, the set H contains exactly n elements (i.e. there are no different indices j and k such that $h(u_j) = h(u_k)$), so with no confusion one can find the index j that fits the equation $\pi(i) = j$. Therefore the condition (4) defines correctly the mapping π.

Now let's check whether π is injective. Let i, $k \in \mathrm{I}$ be (possibly different) indices such that $\pi(i) = j$ and $\pi(k) = j$. Then $\Pi(h(u_i)) = h(u_j) = \Pi(h(u_k))$; because the permutation Π is injective, there must be $h(u_i) = h(u_k)$. Furthermore, as the monomorphism h is injective, there must be $u_i = u_k$, and once is known that in the given unique-valued sequence (1) there are no identical elements, there must be $i = k$ and the transformation $\pi:\mathrm{I} \rightarrow \mathrm{I}$ must be be injective. As the set of indices I is always finite, the injectivity of π implies its bijectivity, i.e. π is really a permutation on the set of indices $\mathrm{I} = \{1, 2,\dots n\}$.

Once we had shown that π is a permutation on the set of indices I, for a pair of indices $i, j \in \mathrm{I}$ for which $\pi(i) = j$, the defining condition (4) for π can be rewritten as:

$$\Pi(h(u_i)) = h(u_j) = h(u_{\pi(i)}) \tag{5}$$

Now, let the mapping h_1: $G \rightarrow h(G)$ be defined by $h_1(x) = h(x)$ for every $x \in G$. This h_1 "inherits" the property of being monomorphism; as its codomain is the image $h(G)$ it is also surjective, i.e. an isomorphism. This isomorphism possesses its inverse isomorphism $h_1^{-1}: h(G) \rightarrow G$ which will be used to transform the following identities:

$$\begin{aligned}
& u_1 * u_2 * \dots \ * u_n = \\
& = h_1^{-1}(h_1(u_1)) * h_1^{-1}(h_1(u_2)) * \dots \ * h_1^{-1}(h_1(u_n)) = \\
& = h_1^{-1}(h_1(u_1) * h_1(u_2) * \dots \ * h_1(u_n)) = \\
& \overset{(3)}{=} h_1^{-1}(\Pi(h_1(u_1)) * \Pi(h_1(u_2)) * \dots \ * \Pi(h_1(u_n))) = \\
& \overset{(5)}{=} h_1^{-1}(h_1(u_{\pi(1)}) * h_1(u_{\pi(2)}) * \dots \ * h_1(u_{\pi(n)})) = \\
& = h_1^{-1}(h_1(u_{\pi(1)})) * h_1^{-1}(h_1(u_{\pi(2)})) * \dots \ * h_1^{-1}(h_1(u_{\pi(n)})) \\
& = u_{\pi(1)} * u_{\pi(2)} * \dots \ * u_{\pi(n)}
\end{aligned} \tag{6}$$

Comparing the equation (6) together with the fact that (1) presents a unique-valued sequence, we easily conclude that π must be the identical permutation ($\pi(i) = i$, for all $i = 1, 2,\dots n$). Substituting this result in (4) we get that $\Pi(h(u_i)) = h(u_i)$ for all $i = 1, 2,\dots n$, which means that Π has to be the identical permutation on the set H.

By contraposition, we conclude that the equation (2) presents the unique-valued sequence $h(u_1)$, $h(u_2)$,… $h(u_n)$ which is a unique-valued sequence of the same length n as the given sequence presented by (1). ■

Now we are interested – are there some other ways to produce different unique-valued sequences of the same length as a given unique-valued sequence? When $G(, {}^{-1}, e)$ is a group, there are three available processes that might be used.

Definition 2. Let u_1, u_2,… u_n be a given unique-valued sequence with value v in the group $G(, {}^{-1}, e)$. Then, the following new sequences of length n can be obtained from it:

(i) u_n^{-1}, u_{n-1}^{-1},… , u_1^{-1} is its ***inverted sequence***;

(ii) v^{-1}, u_1, u_2,… , u_{n-1} is its ***right-shifted sequence***;

(iii) u_2, u_3,… , u_n, v^{-1} is its ***left-shifted sequence***.

These processes of obtaining these new sequences from the given one are to be called (respectively) ***inverting***, ***right-shifting*** and ***left-shifting*** of the initial sequence.

Proposition 3. Let $u_1, u_2, \ldots u_n$ be a unique-valued sequence with value v in the group $G(, {}^{-1}, e)$. Then, the following sequences of length n are unique-valued:

(i) the inverted sequence: $u_n^{-1}, u_{n-1}^{-1}, \ldots, u_1^{-1}$ with value v^{-1};

(ii) the right-shifted sequence: $v^{-1}, u_1, u_2, \ldots, u_{n-1}$ with value u_n^{-1};

(iii) the left-shifted sequence: $u_2, u_3, \ldots, u_n, v^{-1}$ with value u_1^{-1}.

Proof. The proofs of all the three statements will be made by contraposition with the fact that the given sequence is unique-valued with value v, presented by the equation:

$$v = u_1 . u_2 . \ldots . u_n \quad (1)$$

(i) Computing the (multiplicative) inverses of the both sides of (1) we obtain the inverted sequence:

$$v^{-1} = u_n^{-1} . u_{n-1}^{-1} . \ldots . u_1^{-1} \quad (2)$$

Suppose it were not unique-valued; then there exists a non-identical permutation π on the set of indices I = {1, 2,… n} such that:

$$v^{-1} = u_{\pi(n)}^{-1} . u_{\pi(n-1)}^{-1} . \ldots . u_{\pi(1)}^{-1} \quad (3)$$

Again, computing the inverses of both sides of (3) we get:

$$v = u_{\pi(1)} * u_{\pi(2)} * \ldots * u_{\pi(n)}$$

which is contradicted with the facts that (1) presents a unique-valued sequence. So, the inverted sequence must be unique-valued sequence and is presented by (2).

(ii) To avoid complications with the notation, denote $u_0 = v^{-1}$. So (1) becomes:

$$u_0^{-1} = u_1 . u_2 . \ldots . u_n$$

By left multiplying by u_0 of the both sides we get:

$$e = u_0 . u_0^{-1} = u_0 . u_1 . \ldots . u_{n-1} u_n$$

and by right multiplying by u_n^{-1} finally we get:

$$u_n^{-1} = u_0 . u_1 . \ldots . u_{n-1} \quad (4)$$

Until now we're convinced only that u_n^{-1} is the value; but still we don't know whether the right-shifted sequence presented in (4) is really unique-valued. Suppose it were not; then there should exist a non-identical permutation on the set π on the set of indices I_0 = {0, 1,… n-1} such that:

$$u_n^{-1} = u_{\pi(0)} * u_{\pi(1)} * \ldots * u_{\pi(n-1)} \quad (5)$$

Let $i \in I_0$ be the index for which $\pi(i) = 0$. Rewrite (5) as:

$$u_{\pi(0)} * \ldots * u_{\pi(i-1)} * u_{\pi(i)} * u_{\pi(i+1)} * \ldots * u_{\pi(n-1)} = u_n^{-1} \quad (6)$$

Now, the idea is to "dislocate" anything but $u_{\pi(i)} = u_0 = v^{-1}$ from the left side of (6). This is done by left multiplying with the inverses of all elements that lie on the left hand side of $u_{\pi(i)}$:

$$u_{\pi(i)} * u_{\pi(i+1)} * \ldots * u_{\pi(n-1)} = u_{\pi(i-1)}^{-1} * \ldots * u_{\pi(0)}^{-1} * u_n^{-1}$$

and then by right multiplying with the inverses of all elements that lie on the right hand side of $u_{\pi(i)}$:

$$u_{\pi(i)} = u_{\pi(i-1)}^{-1} * \ldots * u_{\pi(0)}^{-1} * u_n^{-1} * u_{\pi(n-1)}^{-1} * \ldots * u_{\pi(i+1)}^{-1}, \quad \text{i.e.}$$

$$v^{-1} = u_{\pi(i-1)}^{-1} * \ldots * u_{\pi(0)}^{-1} * u_n^{-1} * u_{\pi(n-1)}^{-1} * \ldots * u_{\pi(i+1)}^{-1} \tag{7}$$

By inverting (7) one gets:

$$v = u_{\pi(i+1)} * \ldots * u_{\pi(n-1)} * u_n * u_{\pi(0)} * \ldots * u_{\pi(i-1)} \tag{8}$$

Let's observe the factors on the right hand side of (8). The element u_n is obviously a factor, but among the factors can be found all the elements u_1, u_2,… u_{n-1}, each one exactly once (recall that i was chosen such that $\pi(i) = 0$, so indices obtained by π on the right side are contained in the set $\pi(I_0\backslash\{i\}) = I_0\backslash\{0\}=\{1, 2,\ldots n-1\}$). In other words, the right side of (8) is some product of the elements u_1, u_2,… u_n. Since the sequence u_1, u_2,… u_n is unique-valued (as presented by the equation (1)), this is only possible when u_n is the rightmost factor in (8); due to the construction of (8), this is possible only when u_n^{-1} is the leftmost factor in (7), while the last situation is possible only when $u_{\pi(i)}$ is the leftmost factor in (6). However, the leftmost factor in (6) must be $u_{\pi(0)}$, meaning that $u_{\pi(0)} = u_{\pi(i)} = u_0$. Because π is permutation on the set of indices I_0, this is only possible when $\pi(0) = \pi(i) = 0$. This might be used to rewrite the equations (7) and (8) as:

$$v^{-1} = u_n^{-1} * u_{\pi(n-1)}^{-1} * \ldots * u_{\pi(2)}^{-1} * u_{\pi(1)}^{-1} \tag{7'}$$

$$v = u_{\pi(1)} * u_{\pi(2)} * \ldots * u_{\pi(n-1)} * u_n \tag{8'}$$

Now we apply to (8') the discussed fact for (8), that besides the factor u_n, among the other factors on the right side can be found all the elements u_1, u_2,… u_{n-1}, each one exactly once. So, the right side of (8') in fact is a certain product of the elements u_1, u_2,… u_n (arranged in some order) with value v; since the given sequence presented by (1) is unique-valued, the equations (1) and (8') must be identical, which means that $\pi(i) = i$ for all indices $i\in\{1, 2,\ldots n-1\}= I_0\backslash\{0\}$. Together with the already known fact that $\pi(0) = 0$, one gets that $\pi(i) = i$ for all $i\in I_0$, i.e. π is the identical permutation – an obvious contradiction with the choice of π. By the rule of contraposition it follows that the right-shifted sequence presented by (4) is again a unique-valued sequence.

(*iii*) This claim can be proven directly, in completely analogous manner to the proof of (*ii*). However, there is a much shorter proof based on (*i*) and (*ii*): given the unique-valued sequence presented by the equation (1), first by inversion one gets that (2) presents a unique-valued sequence, then by right-shifting:

$$u_1 = v * u_n^{-1} * u_{n-1}^{-1} * \ldots * u_2^{-1} \tag{9}$$

one obtains the unique-valued sequence (9), and eventually by inverting again from (9) one gets that the following equation presents a unique-valued sequence:

$$u_1^{-1} = u_2 * u_3 * \ldots * u_n * v^{-1} \tag{10}$$

which is the presentation of the left-shifted sequence of the given sequence (presented by (1)), what was intended to be proven. ■

How productive are these processes? For a given unique-valued sequence, the inversion of its inversion is the initial sequence again – so this process doesn't seem to be very productive in finding new unique-valued sequences of the same length. However right (or left) shifting can be *repeated* many times (at the n^{th} repetition we get the initial sequence again), which in general seems to be more productive process. The new unique-valued sequences obtained by repeated right (or left) shifting are said to be obtained by "cycling" the original given sequence. For summary, the cycling, inversion and the combination of both are always available processes to produce new unique-valued sequences (not necessarily different from each other and from the original one!) from a given unique-valued sequence in the group $G(, {}^{-1}, e)$.

3. Empirical examination of unique-valued sequences

After the definition and the discovery of the general properties of the unique-valued sequences, it would be very interesting to examine this new concept on well known semigroups. The **recommended problem approach** for a certain semigroup $G(*)$ is the following:

i) First one has to solve the **UPPER BOUND PROBLEM:** "Is there an upper limit L of the *length* of the unique-valued sequences in $G(*)$?"

ii) Then there comes the **EXISTENTIAL PROBLEM:** "Is there a unique-valued sequence of length L in $G(*)$?"

iii) And finally one has to solve the **COMBINATORIAL PROBLEM:** "How many unique-valued sequences there are of all possible lengths $l = 1, 2, \dots, L$ in $G(*)$, when G is finite?"

One of the simplest choices of semigroups on which one can examine the concept of unique-valued sequences, is the family of semigroups of unit matrices of order n.

Definition 3. The square matrix of order n in which the element 1 is positioned on the i-th row and j-th column and all other elements are 0 is to be called ***unit matrix of order*** n and is denoted by E_{ij}.

Note. This name is not used in Wolfram's "MathWorld" [4], the richest on-line mathematical encyclopedia. In matrix types listed in [4] *a*), as well as in the list of integer matrices [4] *b*), these matrices weren't mentioned. However, the term "unit matrix" in [4] *c*) is used to refer another type of integer matrix, which is rarely used. In [1] these matrices are named "*matric units*", which authors of this paper found to be ambiguous and therefore used the term "unit matrices".

The multiplication rule of the unit matrices follows easily from the definition:

$$E_{ij} * E_{kl} = \begin{cases} E_{il}, & \text{if } j = k \\ 0_n, & \text{if } j \neq k \end{cases}$$

where 0_n is the zero matrix of order n.

Definition 4. Let $\mathbf{U}_n()$ denote ***semigroup of unit matrices*** of order n, which is constituted of all the unit matrices of order n together with the zero matrix, where the operation is matrix multiplication.

In our work we are concerned with unique-valued sequences with **non-zero value**. Alternatively, we might have considered the semigroup $\mathbf{U}_n()$ without the zero matrix, while the matrix multiplication should be considered as *partial operation* – the choice we made seems wise, as partial operations are difficult to work with, as one should always check "whether the partial operation *is defined*".

Following the recommended problem approach, first we "attack" the upper bound problem for unique-valued sequences of unit matrices of order n with non-zero value. This is a very hard problem to be proven directly. However, in [1] there is proven the famous **Amitsur-Levitzki Theorem** (which later became the foundation of the theory of "PI-Algebras" = "Polynomial Identity Algebras") which "carries the weight" of that proof. In fact, we only need the Lemma 4 from [1] *a*), which originally states:

„Lemma 4. For each n the standard identity $S_{2n}(x) = 0$ is satisfied by each set of $2n$ units." and using the terminology in this paper it can be rewritten as:

Lemma. For each n and each choice $E^{(1)}, E^{(2)}, \dots, E^{(2n)} \in \mathbf{U}_n$ in the matrix algebra $\mathbf{M}_n(, +, 0_n)$ of order n there is satisfied the standard polynomial identity:

$$\sum_{\pi} \operatorname{sgn}(\pi) \cdot E^{(\pi(1))} * E^{(\pi(2))} * \ldots * E^{(\pi(2n))} = 0_n \tag{1}$$

where in the above sum there are included all the permutations π of the set of indices $\{1, 2, \ldots, 2n\}$, while $\operatorname{sgn}(\pi)$ is the function returning the sign of a permutation: +1 for even and –1 for odd permutation.

As a consequence of this lemma we have:

Proposition 4 (*upper bound problem*). In $\mathbf{U}_n()$ there are no unique-valued sequences of length $2n$ (nor longer).

Proof. Suppose there is a unique-valued sequence $E^{(1)}, E^{(2)}, \ldots, E^{(2n)}$ in $\mathbf{U}_n()$ with length $2n$ and value E_{ij}, which is presented by the following equation:

$$E_{ij} = E^{(1)} . E^{(2)} . \ldots . E^{(2n)} \tag{2}$$

Consider the standard polynomial identity (1), which holds for this sequence too, according to the previous Lemma. When π in (1) becomes the identical permutation, which is even by definition, to the resulting matrix in $\mathbf{M}_n$ is added the matrix E_{ij}, i.e. the value +1 at i-th row and j-th column. However, the Lemma claims that at the end the zero matrix $0_n \in \mathbf{M}_n$ must be obtained; this is possible only when cancellation with –1 has been done, i.e. only when there exists an odd permutation π yielding the same value:

$$E_{ij} = E^{(\pi(1))} . E^{(\pi(2))} . \ldots . E^{(\pi(2n))} \tag{3}$$

The existence of such a permutation π is a contradiction to the assumption that the equation (2) presents a unique-valued sequence. Finally, such a unique-valued sequence of length $2n$ consisted of unit matrices of order n doesn't exist. ■

However, there are such sequences of length $2n - 1$:

Example 2 (*existential problem*). The following equation:

$$E_{1\,n} = E_{11} . E_{12} . E_{22} . E_{23} . \ldots . E_{(n-1)\,n} . E_{n\,n} \tag{4}$$

obviously presents a unique-valued sequence of length $2n - 1$ in $\mathbf{U}_n()$.

Proof. The proof is based on the simple idea – the initial order of multiplication on the right hand side of (4), in fact, is *the only* order of multiplication yielding a non-zero value. So, let's try to find another order of multiplication that yields a non-zero value. First note that E_{11} must be the first factor, as if it is multiplied from right with any other of the given unit matrices, the result is always the zero matrix. Then, E_{12} must be the second factor. For the third factor one may chose between E_{22} and E_{23}; if E_{23} is used there, then somewhere later must be used E_{22}, yielding zero value – again one has no other choice, but to put E_{22} as third and E_{23} as fourth factor. Inductively repeating this discussion one gets that the right hand side of (4) is the only order of multiplication yielding a non-zero value $E_{1\,n}$, so (4) presents a unique-valued sequence. ■

Finally we have the ***combinatorial problem***: how many unique-valued sequences of length l there are in $\mathbf{U}_n()$, for all $l = 1, 2, \ldots, 2n - 1$? The result for $n \le 4$ (carefully calculated by computer program; we also obtained some results for $n > 4$) is displayed in the following table:

order n ⇩ \ length l ⇨	1	2	3	4	5	6	7
1	1						
2	4	6	4				
3	9	24	48	60	30		
4	16	60	192	480	840	840	336

This triangle of numbers, in which every row has exactly $2n - 1$ nonzero values (according to Proposition 4 and Example 2), was first discovered for the sake of this paper and has been registered in the OEIS [2] under the number A114595. Many known sequences from the OEIS appear to be subsequences of this triangle – say, the first column is obviously the sequence n^2. However, for other sequences it is not easy to prove such claims – so instead a proof we now offer just a very interesting conjecture.

Conjecture 1. The number of all unique-valued sequences of unit matrices of order n of the maximal length $2n - 1$ is the n^{th} term of the sequence A001761 in the OEIS.

A001761: 1, 4, 30, 336, 5040, 95040, 2162160,…
$a_n = (2*(n-2))!/(n-1)!$
(this combinatorial sequence first appeared in [3])

Another interesting family of semigroups on which can be examined the concept of unique-valued sequences is the family of the symmetric groups of permutations $\mathbf{S}_n$. For the sake of these examinations, it seems better when the symmetric groups are treated as groups of permutation matrices instead of permutations themselves.

Definition 5. A ***permutation matrix*** of order n is a matrix obtained from the identity matrix of order n by permuting its rows (or columns). Every row and column contains precisely one 1, while other elements are 0.

Definition 6. Let $\mathbf{S}_n$ denote the group constituted by all the permutation matrices of order n, where the group's multiplicative operation is the matrix multiplication; this group is isomorphic to the ***symmetric group of permutations***, so that name will be used.

This family of semigroups proved to be much harder to be examined, than the family of unit matrices. However, with analysis of the empirical results the authors of this paper obtained this conjecture for the upper bound problem:

Conjecture 2 (*upper bound problem*). In $\mathbf{S}_n$ (for $n \geq 2$) there are no unique-valued sequences of length $2n - 2$ (nor longer).

However, the ***existential problem*** is still **OPEN** – there is no even a conjecture nor basic idea how to construct a unique-valued sequence of length $2n - 3$ in $\mathbf{S}_n$. Only the ***combinatorial problem*** has been solved for $n \leq 4$ (again, calculated by a computer program) and the result is displayed in the following table:

order n ⇩ \ length l ⇨	**1**	**2**	**3**	**4**	**5**
1	1				
2	2				
3	6	18	12		
4	24	456	5664	20640	576

This triangle of numbers, in which the first row has one and every other row has been conjectured to have $2n - 3$ nonzero values, was again first discovered for the sake of this paper and has been registered in the OEIS [2] under the number A097635. Again, many known sequences from the OEIS appear to be subsequences of this triangle – say, the first column is obviously the sequence $n!$. For other sequences it is very hard to prove such claims – so again we offer just a very interesting conjecture.

Conjecture 3. The number of all unique-valued sequences of unit matrices of order n of maximal length $2n - 3$ is the n^{th} number of *either one* of the sequences A002860 or A050129 from the OEIS.

A002860: 1, 2, 12, 576, 161280, 8128512001,…
the number of Latin squares of order *n*
A050129: 1, 2, 12, 576, 1658880, 16511297126400,…
$a_1 = 1$, $a_n = n \cdot a^2_n$ for $n>1$.

As a final word from the authors – the exploration of the unique-valued sequences is a new algebraic field (subfield of the PI-algebras) which offers plenty of interesting conjectures to be proven out. Surely, it can also be applied to other non-commutative semigroups (and whole families of semigroups), allowing deeper insight of their structure.

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